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A rationality conjecture about Kontsevich integral of knots and its implications to the structure of the colored Jones polynomial ☆

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Abstract

We formulate a conjecture about the structure of the Kontsevich integral of a knot. We describe its value in terms of the generating functions for the numbers of external edges attached to closed 3-valent diagrams. We conjecture that these functions are rational functions of the exponentials of their arguments, their denominators being the powers of the Alexander–Conway polynomial. This conjecture implies the existence of an expansion of a colored Jones (HOMFLY) polynomial in powers of $q-1$ whose coefficients are rational functions of q^{α} (α being the color assigned to the knot). We show how to derive the first Kontsevich integral polynomial associated to the θ-graph from the rational expansion of the colored $SU(3)$ Jones polynomial.

1. Introduction

The quantum invariants of knots, links and 3-manifolds, such as the Jones polynominal and the Witten–Reshetikhin–Turaev invariant, were discovered about 10 years ago. However, their interpretation in terms of classical 3-dimensional topology still remains a mystery.

Let us compare the skein relation definition of the Jones polynomial to that of a much older Alexander–Conway polynomial. The single-variable Alexander–Conway polynomial $\Delta_{\mathrm{A}}(\mathcal{L};t) \in \mathbb{Z}[t^{\pm 1}]$ is a unique invariant of links in S^3 which satisfies the following two properties. First, the normalization condition:

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Fig. 1. The links $\mathcal{L}_+$, $\mathcal{L}_-$ and $\mathcal{L}_0$.

$$\Delta_{\mathrm{A}}(\text{unknot}; t) = 1. \tag{1.1}$$

Second, if $\mathcal{L}_+$, $\mathcal{L}_-$ and $\mathcal{L}_0$ are three links whose regular projection on a plane is the same except at one spot (see Fig. 1), then

$$\Delta_{\mathrm{A}}(\mathcal{L}_+; t) - \Delta_{\mathrm{A}}(\mathcal{L}_-; t) = \left(t^{1/2} - t^{-1/2}\right)\Delta_{\mathrm{A}}(\mathcal{L}_0; t). \tag{1.2}$$

This definition is purely combinatorial and it is a bit unnatural from the 3-dimensional point of view, since it requires a projection of a link. However, there exist alternative definitions of the Alexander–Conway polynomial of a knot $\mathcal{K}$ which are purely topological. One derives $\Delta_{\mathrm{A}}(\mathcal{K}; t)$ from the structure of the knot group $\pi_1(S^3 \setminus \mathcal{K})$, and the variable t represents the action of the homology $\pi_1/[\pi_1, \pi_1]$ onto the quotient $[\pi_1, \pi_1]/[[\pi_1, \pi_1], [\pi_1, \pi_1]]$, where π_1 is the group of the knot ($\pi_1 = \pi_1(S^3 \setminus \mathcal{K})$). The other definition relates the Alexander polynomial to the Reidemeister torsion of a local system in the knot complement, the variable t being the twist acquired by that system along the meridian of $\mathcal{K}$. From both definitions of $\Delta_{\mathrm{A}}(\mathcal{K}; t)$ it is clear that t is intimately related to the meridian of $\mathcal{K}$.

The Jones polynomial of links $J_2(\mathcal{L}; q) \in \mathbb{Z}[q^{\pm 1/2}]$ can also be defined by skein relations. It is the unique invariant which satisfies the following two properties: the normalization condition

$$J_2(\text{unknot}; q) = q^{1/2} + q^{-1/2} \tag{1.3}$$

and the skein relation

$$q\, J_2(\mathcal{L}_+; q) - q^{-1} J_2(\mathcal{L}_-; q) = \left(q^{1/2} - q^{-1/2}\right) J_2(\mathcal{L}_0; q), \tag{1.4}$$

where the links $\mathcal{L}_+$, $\mathcal{L}_-$ and $\mathcal{L}_0$ are the same as those in Eq. (1.2). Despite an obvious similarity between Eqs. (1.1), (1.2) and Eqs. (1.3), (1.4), there exists not interpretation of $J_2(\mathcal{L}; q)$ in terms of the "classical" objects of 3-dimensional topology, such as the fundamental group of the knot complement. In particular, there is no indication that the variable q has any connection to the meridian of $\mathcal{K}$.

A new hope for a topological interpretation of $J_2(\mathcal{L}; q)$ emerged when J. Birman, X.-S. Lin and D. Bar-Natan discovered that both the Alexander–Conway and Jones polynomials are packed with Vassiliev invariants. Consider the expansions

$$\Delta_{\mathrm{A}}(\mathcal{K}; t) = \sum_{n=0}^{\infty} a_n(\mathcal{K})(t-1)^n, \tag{1.5}$$

$$J_2(\mathcal{K}; q) = \sum_{n=0}^{\infty} b_n(\mathcal{K})(q-1)^n. \tag{1.6}$$

It is not hard to see from the skein relations (1.2) and (1.4) that the coefficients $\alpha_n(\mathcal{K})$ and $\beta_n(\mathcal{K})$ are Vassiliev invariants of degree n. However, Vassiliev invariants by definition are related to the topology of “the space of all maps $S^1 \to S^3$, rather than to the topology of knots themselves. The latter relation is still missing, although some bits of it are known, such as the relation between the tree Vassiliev invariants and Milnor’s linking numbers (see [6] and references therein).

By looking at Eq. (1.5) we may say that the Alexander–Conway polynomial presents a way of assembling some Vassiliev invariants of knots into a polynomial which has a clear interpretation in terms of the classical 3-dimensional topology. At the same time, the Jones polynomial assembles some other Vassiliev invariants into another polynomial whose topological origin is rather obscure. Therefore one may wonder if there is a way of reassembling all Vassiliev invariants into the polynomials which would be similar to the Alexander–Conway polynomial rather than to the Jones polynomial in terms of their topological interpretation.

In Sections 2 and 3 we present an algorithm of assembling Vassiliev invariants coming from the Kontsevich integral of a knot into a sequence of functions of a variable t. In Section 4 we conjecture that these functions are rational: their denominators are powers of the Alexander–Conway polynomial of t while their numerators are new polynomial invariants of knots. Since these new polynomials depend on the same variable t, we expect them to have a topological interpretation in which, similarly to the case of the Alexander–Conway polynomial, t will also be related to the meridian of a knot.

Since the first version of this paper was written and reported, Andrew Kricker has proved the rationality conjecture in his paper [7].

Kontsevich integral is related to the colored Jones (HOMFLY) polynomial of the knot through the application of a Lie algebra weight system. In Section 5 we explain how to apply this weight system to the ‘repackaged’ Vassiliev invariants. Then we show how the rational structure of Kontsevich integral appears as a rational the Jones polynomial. In Section 6 we use these results to extract the first non-trivial knot polynomial related to the θ-graph from the expansion of the $SU(3)$ colored Jones polynomial. In Appendix A we present a table of these ‘2-loop’ polynomials for knots with up to 7 crossings.

2. Graph spaces

We are going to define an algebra $\mathcal{D}$ based on 3-valent graphs, but first let us recall the definition of the algebra $\mathcal{B}$ of $(1,3)$-valent graphs related to Vassiliev invariants of a knot. Each 3-valent vertex of a graph is endowed with a cyclic ordering of 3 egdes attached to it. When we draw a picture of a graph, we assume that this ordering is counterclockwise.

A graph D has 2 degrees. They are defined as

$$\deg_1(D) = \#\text{1-vertices}, \tag{2.1}$$

$$\deg_2(D) = \#\text{chords} - \#\text{3-vertices} = \chi(D) + \deg_1(D), \tag{2.2}$$

where $\chi(D)$ is the Euler characteristic of D (more precisely, $\chi(D)$ denotes the Euler characteristic with the *opposite* sign).

Let $\widetilde{\mathcal{B}}_{m,n}$ be a formal vector space (over $\mathbb{C}$) whose basis elements are in a one-to-one correspondence with $(1,3)$-valent graphs of degrees m and n

$$\widetilde{\mathcal{B}}_{m,n} = \operatorname{span}\bigl(D \mid \deg_1(D) = m, \deg_2(D) = n\bigr). \tag{2.3}$$

Together all such spaces form a graded space $\widetilde{\mathcal{B}}$

$$\widetilde{\mathcal{B}} = \bigoplus_{n=0}^{\infty} \widetilde{\mathcal{B}}_n, \quad \text{where } \widetilde{\mathcal{B}}_n = \bigoplus_{m=0}^{\infty} \widetilde{\mathcal{B}}_{m,n}. \tag{2.4}$$

The space $\widetilde{\mathcal{B}}$ has two important subspaces: $\widetilde{\mathcal{B}}_{\mathrm{AS}}$ and $\widetilde{\mathcal{B}}_{\mathrm{IHX}}$. $\widetilde{\mathcal{B}}_{\mathrm{AS}}$ is spanned by the sums $D_1 + D_2$, where D_1 and D_2 are the same graphs except that they have different cyclic orders at one 3-valent vertex:

$$\widetilde{\mathcal{B}}_{\mathrm{AS}} = \operatorname{span}(D_1 + D_2 \text{ for all pairs } D_1, D_2). \tag{2.5}$$

In order to define $\widetilde{\mathcal{B}}_{\mathrm{IHX}}$, consider a space $\widetilde{\mathcal{B}}'$ whose basis vectors are graphs with 1-valent and 3-valent vertices and exactly one 4-valent vertex. We define a linear map $\partial_{\mathrm{IHX}} : \widetilde{\mathcal{B}}' \to \widetilde{\mathcal{B}}$ by its action on the individual graphs of $D \in \widetilde{\mathcal{B}}'$

$$\partial_{\mathrm{IHX}} : D \mapsto D_1 - D_2 + D_3, \tag{2.6}$$

where all four graphs D, D_1, D_2, D_3 are the same except at one spot, where they differ according to Fig. 2. Then we define the second subspace $\widetilde{\mathcal{B}}_{\mathrm{IHX}} \subset \widetilde{\mathcal{B}}$ as the image of ∂_{IHX}.

Now we introduce a space

$$\mathcal{B} = \widetilde{\mathcal{B}} / \bigl(\widetilde{\mathcal{B}}_{\mathrm{AS}} + \widetilde{\mathcal{B}}_{\mathrm{IHX}}\bigr). \tag{2.7}$$

Since the graphs D_1, D_2 of (2.5) and the graphs D_1, D_2, D_3 of (2.6) have the same degrees (2.1), (2.2) among themselves, then both subspaces $\widetilde{\mathcal{B}}_{\mathrm{AS}}$ and $\widetilde{\mathcal{B}}_{\mathrm{IHX}}$ respect the gradings (2.4) and as a result the space $\mathcal{B}$ is also graded

$$\mathcal{B} = \bigoplus_{n=0}^{\infty} \mathcal{B}_n, \quad \text{where } \mathcal{B}_n = \bigoplus_{m=0}^{\infty} \mathcal{B}_{m,n}. \tag{2.8}$$

It is well-known that the dual space $\mathcal{B}^*$ is isomorphic to the space of all Vassiliev invariants of knots, and the grading $\mathcal{B}^* = \bigoplus_{n=0}^{\infty} \mathcal{B}_n^*$ corresponds to the grading of Vassiliev invariants.

The space $\mathcal{B}$ can be endowed with a commutative algebra structure. We define the product of two graphs in $\widetilde{\mathcal{B}}$ as their disjoint union. It is easy to see that this product respects the gradings (2.1), (2.2) and that the subspace $\widetilde{\mathcal{B}}_{\mathrm{AS}} + \widetilde{\mathcal{B}}_{\mathrm{IHX}}$ is the ideal in algebra $\widetilde{\mathcal{B}}$. Therefore, the quotient space $\mathcal{B}$ is also an algebra.

We are going to introduce another algebra $\mathcal{D}$ which is isomorphic to $\mathcal{B}$. This construction has been known to some people [1]. It appeared as an attempt to better understand the structure of $\mathcal{B}$ and, in particular, to evaluate the dimension of the spaces $\mathcal{B}_n$.

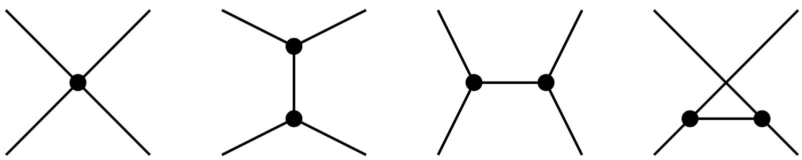

Fig. 2. The graphs D, D_1, D_2 and D_3.

I am especially indebted to A. Vaintrob for illuminating discussions on the structure of $\mathcal{D}$. I introduce the algebra $\mathcal{D}$ in order to formulate a conjecture about the structure of Kontsevich integral, which was motivated by the study of the Melvin–Morton expansion of the colored Jones polynomial as it comes of R-matrix expression and which has now been proved by Kricker [7].

We begin by defining a bigger space $\widetilde{\mathcal{D}}$. Let D be a graph with 3-valent vertices and no 1-valent vertices. We think of this graph as a CW-complex and consider a space of its rational cohomologies $H^1(D,\mathbb{Q})$. Let G_D be the group of symmetry of a graph D (it maps 3-vertices to 3-vertices and edges to edges) and let $G_D^\star \subset G_D$ be its subgroup which preserves the cyclic order of the edges at the vertices. G_D acts naturally on $H^1(D,\mathbb{Q})$ and this group action can be extended to the symmetric algebra $S^*H^1(D,\mathbb{Q})$. We denote by $\mathcal{H}^\star(D)$ the $G_D^\star$-invariant part of the latter space:

$$\mathcal{H}^\star(D) = \bigoplus_{m=0}^{\infty} \mathcal{H}^\star_m(D), \quad \mathcal{H}^\star_m(D) = \left(S^m H^1(D,\mathbb{Q})\right)_{G_D^\star}, \tag{2.9}$$

while $P_D^\star$ is the corresponding projector

$$P_D^\star : S^*H^1(D,\mathbb{Q}) \to \mathcal{H}^\star(D), \quad P_D^\star(x) = \frac{1}{|G_D^\star|} \sum_{g\in G_D^\star} g(x), \tag{2.10}$$

where $|G_D^\star|$ denotes the number of elements in $G_D^\star$. Now we define a linear space $\widetilde{\mathcal{D}}^\star$ as

$$\widetilde{\mathcal{D}}^\star = \bigoplus_{m,n=0}^{\infty} \widetilde{\mathcal{D}}^\star_{m,n}, \quad \text{where } \widetilde{\mathcal{D}}^\star_{m,n} = \bigoplus_{D:\ \chi(D)=n} \mathcal{H}^\star_m(D). \tag{2.11}$$

The space $\widetilde{\mathcal{D}}^\star$ has an associative, commutative algebra structure. First, note that for a disjoint union $D_1 \cup D_2$ of two graphs D_1, D_2

$$H^1(D_1 \cup D_2,\mathbb{Q}) = H^1(D_1,\mathbb{Q}) \oplus H^1(D_2,\mathbb{Q}) \tag{2.12}$$

and therefore

$$S^*H^1(D_1 \cup D_2,\mathbb{Q}) = S^*H^1(D_1,\mathbb{Q}) \otimes S^*H^1(D_2,\mathbb{Q}) \tag{2.13}$$

as algebras. The latter equation allows us to define a product of two elements $x_i \in H^1(D_i,\mathbb{Q})$, $i = 1, 2$, as a projection of their tensor product in $S^*H^1(D_1 \cup D_2,\mathbb{Q})$

$$x_1x_2 = P^\star_{D_1\cup D_2}(x_1 \otimes x_2) \in \mathcal{H}^\star(D_1 \cup D_2). \tag{2.14}$$

If the graphs D_1, D_2 do not have isomorphic connected components, then $G^\star_{D_1\cup D_2} = G^\star_{D_1} \times G^\star_{D_2}$ and the projector in Eq. (2.14) may be omitted: $x_1x_2 = x_1 \otimes x_2$. The commutativity of the product (2.14) is obvious. Associativity follows from a relation

$$(x_1x_2)x_3 = x_1(x_2x_3) = P^\star_{D_1\cup D_2\cup D_3}(x_1 \otimes x_2 \otimes x_3). \tag{2.15}$$

Finally, since the product (2.14) respects both gradings (2.11), then $\widetilde{\mathcal{D}}^\star$ is a graded algebra.

Next, we define the subspace $\widetilde{\mathcal{D}}^\star_{\mathrm{AS}} \subset \widetilde{\mathcal{D}}^\star$ which comes from the change of cyclic order at 3-valent vertices. The definition of the symmetric algebra $S^*H^1(D,\mathbb{Q})$ is independent of this cyclic order. Therefore if we take a graph D_1 and change the cyclic order at one of its vertices, thus producing a new graph D_2, then there is a natural isomorphism of cohomologies

$$\hat{f}_{\mathrm{AS}} : H^1(D_1, \mathbb{Q}) \to H^1(D_2, \mathbb{Q}), \tag{2.16}$$

because D_2 was constructed in such a way that there is a natural isomorphism between D_1 and D_2 as CW-complexes (generally, there could be more than one isomorphism due to the symmetry group G_{D_1}). The isomorphism (2.16) can be extended to an isomorphism of symmetric algebras

$$\hat{f}_{\mathrm{AS}} : S^* H^1(D_1, \mathbb{Q}) \to S^* H^1(D_2, \mathbb{Q}), \tag{2.17}$$

let $\widetilde{V}_{\mathrm{AS}}$ be the graph of this map

$$\widetilde{V}_{\mathrm{AS}} = \bigl\{(x, y) \mid y = \hat{f}_{\mathrm{AS}}(x)\bigr\} \subset S^* H^1(D_1, \mathbb{Q}) \oplus S^* H^1(D_2, \mathbb{Q}). \tag{2.18}$$

We denote by V_{AS} its projection onto $\mathcal{H}^\star(D_1) \oplus \mathcal{H}^\star(D_2)$

$$V_{\mathrm{AS}} = P^\star_{D_1} P^\star_{D_2}\bigl(\widetilde{V}_{\mathrm{AS}}\bigr) \subset \mathcal{H}^\star(D_1) \oplus \mathcal{H}^\star(D_2). \tag{2.19}$$

We define the subspace $\widetilde{\mathcal{D}}^\star_{\mathrm{AS}} \subset \widetilde{\mathcal{D}}^\star$ as the sum of all the spaces V_{AS} for all 3-valent diagrams D_1 and all choices of vertices of D_1 where we change the orientation. It is easy to check that $\widetilde{\mathcal{D}}^\star_{\mathrm{AS}}$ is an ideal: for any element $x \in S^* H^1(D_1, \mathbb{Q})$, and for any element $y \in \mathcal{H}^\star(D_3)$

$$\begin{aligned}\bigl(P^\star_{D_1}(x) + P^\star_{D_2} \hat{f}_{\mathrm{AS}}(x)\bigr)y &= P^\star_{D_1 \cup D_3}\bigl(P^\star_{D_1}(x) \otimes y\bigr) + P^\star_{D_2 \cup D_3}\bigl(P^\star_{D_2} \hat{f}_{\mathrm{AS}}(x) \otimes y\bigr)\\ &= P^\star_{D_1 \cup D_3}(x \otimes y) + P^\star_{D_2 \cup D_3} \hat{f}_{\mathrm{AS}}(x \otimes y),\end{aligned} \tag{2.20}$$

because obviously

$$P^\star_{D_i \cup D_3}\bigl(P^\star_{D_i} \otimes I\bigr) = P^\star_{D_i \cup D_3}, \quad i = 1, 2, \tag{2.21}$$

and

$$\hat{f}_{\mathrm{AS}}(x \otimes y) = \hat{f}_{\mathrm{AS}}(x) \otimes y, \tag{2.22}$$

where in the l.h.s. $\hat{f}_{\mathrm{AS}}$ comes from the change of cyclic order at a vertex in the whole graph $D_1 \cup D_3$.

Finally, we define a subspace $\widetilde{\mathcal{D}}^\star_{\mathrm{IHX}} \subset \widetilde{\mathcal{D}}^\star$. Let D be a graph with 3-valent vertices and exactly one 4-valent vertex, and with fixed cyclic order at every vertex. By adding an extra edge to D, we "resolve" the 4-valent vertex in 3 different ways, thus converting D into one of the 3-valent graphs D_1, D_2, D_3 of Fig. 2. A removal of this extra edge generates 3 natural maps of rational homologies

$$\hat{f}_i : H_1(D_i, \mathbb{Q}) \to H_1(D, \mathbb{Q}), \quad i = 1, 2, 3. \tag{2.23}$$

We extend the dual maps $\hat{f}^*_i : H^1(D, \mathbb{Q}) \to H^1(D_i, \mathbb{Q})$ as algebra homomorphisms

$$\hat{f}^*_i : S^* H^1(D, \mathbb{Q}) \to S^* H^1(D_i, \mathbb{Q}), \quad i = 1, 2, 3. \tag{2.24}$$

We define the map $\hat{\partial}_{\mathrm{IHX}} : S^* H^1(D, \mathbb{Q}) \to \bigoplus_{i=1}^3 \mathcal{H}^\star(D_i)$ by the formula (cf. Eq. (2.6))

$$\hat{\partial}_{\mathrm{IHX}} = -P^{\star}_{D_1}\hat{f}_1^* + P^{\star}_{D_2}\hat{f}_2^* - P^{\star}_{D_3}\hat{f}_3^*. \tag{2.25}$$

The subspace $\widetilde{\mathcal{D}}^{\star}_{\mathrm{IHX}}$ is the sum of the images of all the maps $\hat{\partial}_{\mathrm{IHX}}$ for all the graphs D. It is easy to check that similarly to $\widetilde{\mathcal{D}}^{\star}_{\mathrm{AS}}$, $\widetilde{\mathcal{D}}^{\star}_{\mathrm{IHX}}$ is also an ideal in $\widetilde{\mathcal{D}}^{\star}$.

Now we define the quotient space

$$\mathcal{D} = \widetilde{\mathcal{D}}^{\star}/\big(\widetilde{\mathcal{D}}^{\star}_{\mathrm{AS}} + \widetilde{\mathcal{D}}^{\star}_{\mathrm{IHX}}\big). \tag{2.26}$$

Since the graphs D_1, D_2 of (2.16) and D, D_1, D_2, D_3 of (2.23) have the same Euler characteristic among themselves and since the maps (2.17) and (2.24) preserve the grading of symmetric algebras, then $\mathcal{D}$ is a graded algebra:

$$\mathcal{D} = \bigoplus_{m,n=0}^{\infty} \mathcal{D}_{m,n}, \tag{2.27}$$

where the spaces $\mathcal{D}_{m,n}$ are the quotients of the spaces $\widetilde{\mathcal{D}}_{m,n}$.

$$\mathcal{D}_{m,n} = \widetilde{\mathcal{D}}_{m,n}/\mathcal{D}_{m,n} \cap \big(\widetilde{\mathcal{D}}_{\mathrm{AS}} + \widetilde{\mathcal{D}}_{\mathrm{IHX}}\big). \tag{2.28}$$

This description of the algebra $\mathcal{D}$ makes it easy to establish its isomorphism with the algebra $\mathcal{B}$, but there exists a slightly different description of $\mathcal{D}$ which suits better for the formulation of our conjecture about the structure of Kontsevich integral. Recall that G_D denotes the full symmetry group of a 3-valent graph D (including the maps which do not preserve the cyclic order at the vertices). As we have mentioned, G_D acts naturally on $S^*H^1(D,\mathbb{Q})$. We modify this action by multiplying the action of an element $g \in G_D$ by $(-1)^{|g|}$, where $|g|$ denotes the number of vertices of D whose cyclic order is changed by g. Now instead of (2.9) we define

$$\mathcal{H}(D) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m(D), \quad \mathcal{H}_m(D) = \big(S^m H^1(D,\mathbb{Q})\big)_{G_D}, \tag{2.29}$$

while P_D is the corresponding projector

$$P_D : S^*H^1(D,\mathbb{Q}) \to \mathcal{H}(D), \quad P_D(x) = \frac{1}{|G_D|}\sum_{g\in G_D} g(x). \tag{2.30}$$

Let $\boldsymbol{D}$ be a set of all 3-valent graphs with a particular cyclic order of edges at vertices chosen for every graph (so that each isomorphism class of 3-valent graphs is represented in $\boldsymbol{D}$ exactly once). Define

$$\widetilde{\mathcal{D}} = \bigoplus_{m,n=0}^{\infty} \widetilde{\mathcal{D}}_{m,n}, \quad \text{where } \widetilde{\mathcal{D}}_{m,n} = \bigoplus_{D:\ \chi(D)=n} \mathcal{H}_m(D) \tag{2.31}$$

(cf. Eq. (2.11)). If we choose a different set $\boldsymbol{D}'$, then there is a natural isomorphism between $\widetilde{\mathcal{D}}_{\boldsymbol{D}}$ and $\widetilde{\mathcal{D}}_{\boldsymbol{D}'}$. Namely, if $D_1 \in \boldsymbol{D}$ and $D_2 \in \boldsymbol{D}'$ represent the same 3-valent graph (but possibly with different cyclic orders), then we identify the spaces $\mathcal{H}(D_1)$ and $\mathcal{H}(D_2)$ by an identity map with an extra sign factor $(-1)^{\#(D_1,D_2)}$, where $\#(D_1,D_2)$ is the number of vertices in the graphs D_1, D_2 which have different cyclic orders. In the future we will sometimes denote $\widetilde{\mathcal{D}}_{\boldsymbol{D}}$ simply as $\widetilde{\mathcal{D}}$, assuming that the choice of cyclic order for every 3-valent graph was somehow fixed.

Lemma 2.1. *There is a natural isomorphism* $\widetilde{\mathcal{D}}_{\boldsymbol{D}} \cong \widetilde{\mathcal{D}}^\star/\widetilde{\mathcal{D}}^\star_{\mathrm{AS}}$.

Proof. Since $G^\star_D \subset G_D$, then $\mathcal{H}(D) \subset \mathcal{H}^\star(D)$. As a result, $\widetilde{\mathcal{D}}_{\boldsymbol{D}}$ may be considered a subspace of $\widetilde{\mathcal{D}}^\star$ and thus we have a map $f:\widetilde{\mathcal{D}}_{\boldsymbol{D}} \to \widetilde{\mathcal{D}}^\star/\widetilde{\mathcal{D}}^\star_{\mathrm{AS}}$. On the other hand, one can construct a natural map $g:\widetilde{\mathcal{D}}^\star \to \widetilde{\mathcal{D}}_{\boldsymbol{D}}$ in the following way: if a 3-valent graph D_1 is isomorphic to a graph $D_2 \in \boldsymbol{D}$, then g maps $\mathcal{H}^\star(D_1)$ to $\mathcal{H}(D_2) \subset \mathcal{H}^\star(D_1)$ as $(-1)^{\#(D_1,D_2)} P_{D_1}$. Obviously, $\widetilde{\mathcal{D}}^\star_{\mathrm{AS}} \subset \ker g$, so we have a map $h:\widetilde{\mathcal{D}}^\star/\widetilde{\mathcal{D}}^\star_{\mathrm{AS}} \to \widetilde{\mathcal{D}}_{\boldsymbol{D}}$. We leave it to the reader to check that f and h constitute an isomorphism. $\square$

After constructing an isomorphism $h:\widetilde{\mathcal{D}}^\star/\widetilde{\mathcal{D}}^\star_{\mathrm{AS}} \to \widetilde{\mathcal{D}}_{\boldsymbol{D}}$ we define the space $\widetilde{\mathcal{D}}_{\mathrm{IHX}}$ simply as the image of $\widetilde{\mathcal{D}}^\star_{\mathrm{IHX}}/(\widetilde{\mathcal{D}}^\star_{\mathrm{IHX}} \cap \widetilde{\mathcal{D}}^\star_{\mathrm{AS}})$. Thus we proved the following

Theorem 2.2. *There is a natural isomorphism* $\mathcal{D} \cong \widetilde{\mathcal{D}}_{\boldsymbol{D}}/\widetilde{\mathcal{D}}_{\mathrm{IHX}}$.

The grading subspaces $\mathcal{D}_{m,n}$ turn out to be the quotients $\widetilde{\mathcal{D}}_{m,n}/(\widetilde{\mathcal{D}}_{m,n} \cup \widetilde{\mathcal{D}}_{\mathrm{IHX}})$.

The advantage of this description of $\mathcal{D}$ is that it allows us to work with rather natural spaces $(S^m H^1(D,\mathbb{Q}))_{G_D}$ instead of bigger and less symmetric spaces $(S^m H^1(D,\mathbb{Q}))_{G^\star_D}$.

3. Isomorphism between $\mathcal{B}$ and $\mathcal{D}$

Theorem 3.1. *There exists a canonical isomorphism of algebras*

$$\widehat{A}:\mathcal{B} \to \mathcal{D}, \tag{3.1}$$

which respects the grading

$$\widehat{A}:\mathcal{B}_{m,n} \to \mathcal{D}_{m,n-m}. \tag{3.2}$$

Corollary 3.2. *If* $m > n$, *then* $\mathcal{B}_{m,n} = \emptyset$.

Before we prove this theorem, we have to establish some facts concerning the structure of the space $\mathcal{B}$. We call an edge of a $(1,3)$-valent graph *a leg* if this edge is connected to a 1-valent vertex. All other edges are called *internal*.

Lemma 3.3. *If two legs of a* $(1,3)$*-valent graph D are attached to the same* 3*-valent vertex, then* $D \in \widetilde{\mathcal{B}}_{\mathrm{AS}}$.

Proof. Suppose that a $(1,3)$-valent graph D contains such a 3-valent vertex. Since the 1-valent vertices of our graphs are not ordered in any way, then changing the cyclic order at that 3-valent vertex does not change the graph. Therefore $2D \in \widetilde{\mathcal{B}}_{\mathrm{AS}}$ and this proves the lemma. $\square$

Let us call a $(1,3)$-valent graph *restricted* if each of its 3-valent vertices contains at most one leg. Let $\widetilde{\mathcal{B}}^{(\mathrm{r})}$ be a formal space whose basis vectors are restricted graphs. We introduce familiar subspaces. The subspaces $\widetilde{\mathcal{B}}^{(i)}_{\mathrm{AS}} \subset \widetilde{\mathcal{B}}^{(\mathrm{r})},\ i = 0, 1,$ are spanned by the sums of restricted diagrams D_1, D_2 which differ in the ordering at a 3-valent vertex which is attached to i legs. The subspaces $\widetilde{\mathcal{B}}^{(i)}_{\mathrm{IHX}} \subset \widetilde{\mathcal{B}}^{(\mathrm{r})},\ i = 0, 1,$ are spanned by the images of the map (2.6) acting on the $(3, 4)$-valent diagrams whose single 4-valent vertex contains i legs. Then Lemma 3.3 has a simple corollary:

$$\mathcal{B}_{m,n} = \widetilde{\mathcal{B}}^{(\mathrm{r},0)}_{m,n} / \bigl(\widetilde{\mathcal{B}}^{(0)}_{\mathrm{AS}} + \widetilde{\mathcal{B}}^{(0)}_{\mathrm{IHX}}\bigr), \quad \text{where } \widetilde{\mathcal{B}}^{(\mathrm{r},0)}_{m,n} = \widetilde{\mathcal{B}}^{(\mathrm{r})}_{m,n} / \bigl(\widetilde{\mathcal{B}}^{(1)}_{\mathrm{AS}} + \widetilde{\mathcal{B}}^{(1)}_{\mathrm{IHX}}\bigr). \tag{3.3}$$

Indeed, this relation follows from the fact that if the 4-valent vertex of a $(3, 4)$-valent graph D has at least two legs, then the intersection of the image of the corresponding operator (2.6) with the space $\widetilde{\mathcal{B}}^{(\mathrm{r})}$ is trivial. Also, it is easy to see that $\widetilde{\mathcal{B}}^{(1)}_{\mathrm{AS}}$ and $\widetilde{\mathcal{B}}^{(1)}_{\mathrm{IHX}}$ are ideals in $\widetilde{\mathcal{B}}^{(\mathrm{r})}$, so the quotient

$$\widetilde{\mathcal{B}}^{(\mathrm{r},0)} = \bigoplus_{m,n=0}^{\infty} \widetilde{\mathcal{B}}^{(\mathrm{r},0)}_{m,n} = \widetilde{\mathcal{B}}^{(\mathrm{r})} / \bigl(\widetilde{\mathcal{B}}^{(1)}_{\mathrm{AS}} + \widetilde{\mathcal{B}}^{(1)}_{\mathrm{IHX}}\bigr) \tag{3.4}$$

has a graded algebra structure.

Now we begin to construct the isomorphism. Let D be a 3-valent graph with N edges and cyclic order at vertices. Thinking of D as a CW-complex, let C_1 be the space of 1-chains. In other words, C_1 is an N-dimensional vector space spanned by the oriented edges of D, if we assume that an edge with the opposite orientation is equal to the opposite of the edge as an element of C_1. Thus, if we pick an orientation on the edges of D, then C_1 has a natural basis e_j, $1 \leqslant j \leqslant N$, of the edges of D. We will also need the dual space C_1^* with the dual basis f_j, $1 \leqslant j \leqslant N$. The symmetry group of the graph G_D acts on both spaces C_1 and C_1^*.

Next, consider a vector space whose basis is formed by m-legged $(1, 3)$-valent restricted graphs such that if we remove their legs, then we get the 3-valent graph D. We denote the quotient of this space by its intersection with $\widetilde{\mathcal{B}}^{(1)}_{\mathrm{AS}}$ as $\widetilde{\mathcal{B}}_m(D)$. We also have to consider a bigger space. Suppose that we index the edges of D and then attach m legs to its edges in order to produce restricted graphs. These $(1, 3)$-valent graphs still carry the indexing of the edges of D. If we factor this space by its intersection with the obvious analog of $\widetilde{\mathcal{B}}^{(1)}_{\mathrm{AS}}$, then we get the space $\check{\mathcal{B}}_m(D)$. The symmetry group $G^{\star}_D$ of the graph D acts on $\check{\mathcal{B}}_m(D)$ by mapping the edges of D together with their legs, while preserving the cyclic order at the vertices. The invariant subspace of this action is canonically isomorphic to $\widetilde{\mathcal{B}}_m(D)$:

$$\widetilde{\mathcal{B}}_m(D) = \bigl(\check{\mathcal{B}}_m(D)\bigr)_{G^{\star}_D}. \tag{3.5}$$

Let us introduce a multi-index notation

$$\underline{m} = (m_1, \ldots, m_N), \quad |\underline{m}| = \sum_{j=1}^{N} m_j. \tag{3.6}$$

For N non-negative numbers $\underline{m}$ and for a choice of orientation of the edges of D construct a diagram $D_{\underline{m}}$ in the following way: for every j, $1 \leqslant j \leqslant N$, attach m_j legs to D on the left side of the edge e_j (the notion of the left side is well-defined since e_j is oriented). It is easy to see that all graphs $D_{\underline{m}}$, $|\underline{m}| = m$, form a basis of the space $\check{\mathcal{B}}_m(D)$, because after we took the quotient over the analog of the space $\widetilde{\mathcal{B}}^{(1)}_{\mathrm{AS}}$, we can flip the legs of the graphs of $\check{\mathcal{B}}_m(D)$ to a particular side of each edge of D (at the cost of changing the signs of the corresponding vectors of $\check{\mathcal{B}}_m(D)$).

There is a natural isomorphism $A:\check{\mathcal{B}}_m(D)\to S^m C_1^*$ which acts on the basis vectors as

$$\widehat{A}: D_{\underline{m}} \mapsto \prod_{j=1}^{N} f_j^{m_j}. \tag{3.7}$$

Suppose that the 3-valent graph D has N_0 vertices v_j, $1\leqslant j\leqslant N_0$. Consider the N_0-dimensional space C_0 of 0-chains whose basis vectors are in a one-to-one correspondence with these vertices. Then there is a natural boundary map $\partial:C_1\to C_0$. Let $\check{C}_1^*$ be the space of 1-cocycles, it is the subspace of C_1^* whose elements annihilate the kernel of ∂. Apparently,

$$H^1(D,\mathbb{Q}) = C_1^*/\check{C}_1^*. \tag{3.8}$$

Let $\check{\mathcal{B}}^{(1)}_{\mathrm{IHX}}(D,m) = \check{\mathcal{B}}_m(D)\cap\check{\mathcal{B}}^{(1)}_{\mathrm{IHX}}$, where the space $\check{\mathcal{B}}^{(1)}_{\mathrm{IHX}}$ is the analog of the space $\widetilde{\mathcal{B}}^{(1)}_{\mathrm{IHX}}$ for the graphs which come from 3-valent graphs with indexed edges.

Lemma 3.4. *The map $\widehat{A}$ establishes an isomorphism between the spaces $\check{\mathcal{B}}^{(1)}_{\mathrm{IHX}}(D,1)$ and $\check{C}_1^*$.*

Proof. For $1\leqslant j\leqslant N_0$, denote as V_j the image in $\check{\mathcal{B}}_1(D)$ of the operator (2.6) associated with the vertex v_j of D (that is, one of the two 3-valent vertices in each of the graphs of Fig. 2 is v_j, while the other vertex is attached to a leg). Then the space $\check{\mathcal{B}}^{(1)}_{\mathrm{IHX}}(D,1)$ is spanned by all the spaces V_j.

For $1\leqslant j\leqslant N_0$ and for $x\in C_1$ let $\partial_j(x)$ be the coefficient in front of $v_j\in C_0$ in the expansion of $\partial(x)$ with respect to the basis v. Then $\ker\partial=\bigcap_{j=1}^{N_0}\ker\partial_j$ and, as a result, the space $\check{C}_1^*$ is spanned by the spaces $V_j'\subset C_1^*$ which annihilate the spaces $\ker\partial_j\subset C_1$. It is very easy to see that for every j, $\widehat{A}$ establishes an isomorphism between the corresponding spaces V_j and V_j'. This proves the lemma. □

Lemma 3.5. *$\widehat{A}$ establishes the isomorphism between the spaces $\check{\mathcal{B}}_m(D)/\check{\mathcal{B}}^{(1)}_{\mathrm{IHX}}(D,m)$ and $S^m H^1(D,\mathbb{Q})$.*

To prove this lemma we need a simple fact from linear algebra.

Lemma 3.6. *Let V be a finite-dimensional vector space and W be its subspace. Denote by P_{S} a symmetrizing projector $P_{\mathrm{S}}:V^{\otimes m}\to S^m V$. Then*

$$S^m V/P_{\mathrm{S}}\big(S^{m-1}V\otimes W\big) = S^m(V/W). \tag{3.9}$$

Proof. We leave the proof to the reader.

Proof of Lemma 3.5. It is easy to see that $\widehat{A}$ maps the space $\check{\mathcal{B}}^{(1)}_{\mathrm{IHX}}(D,m)$ onto $P_{\mathrm{S}}(S^{m-1}C_1^* \otimes \check{C}_1^*)$. Then the claim of the lemma follows from Eqs. (3.8) and (3.9) if we set $V = C_1^*$ and $W = \check{C}_1^*$ in the latter equation. □

Consider a space $\widetilde{\mathcal{B}}^{(1)}_{\mathrm{IHX}}(D,m) = \widetilde{\mathcal{B}}_m(D) \cap \widetilde{\mathcal{B}}^{(1)}_{\mathrm{IHX}}$.

Lemma 3.7. *There is a natural isomorphism between the quotient spaces*

$$\widetilde{\mathcal{B}}_m(D)/\widetilde{\mathcal{B}}^{(1)}_{\mathrm{IHX}}(D,m) = \big(\check{\mathcal{B}}_m(D)/\check{\mathcal{B}}^{(1)}_{\mathrm{IHX}}(D,m)\big)_{G^\star_D}. \tag{3.10}$$

In order to prove this isomorphism we need another linear algebra lemma.

Lemma 3.8. *Let V be a finite-dimensional representation of a finite group G. Let $W \subset V$ be a subspace, which is invariant under the action of G. Then there is a natural isomorphism*

$$(V)_G/(W)_G = (V/W)_G. \tag{3.11}$$

Proof. For example, one could use the fact that a finite-dimensional representation of G is a sum of irreducible representations. We leave the details to the reader. □

Proof of lemma 3.7. The cyclic order preserving symmetry group $G^\star_D$ of the 3-valent graph D acts on the space $\check{\mathcal{B}}_m(D)$. Obviously, the symmetrization over this action projects $\check{\mathcal{B}}_m(D)$ onto $\widetilde{\mathcal{B}}_m(D)$. Thus

$$\widetilde{\mathcal{B}}_m(D) = \big(\check{\mathcal{B}}_m(D)\big)_{G^\star_D}. \tag{3.12}$$

At the same time, the subspace $\check{\mathcal{B}}^{(1)}_{\mathrm{IHX}}(D,m)$ is invariant under the action of $G^\star_D$ and

$$\widetilde{\mathcal{B}}^{(1)}_{\mathrm{IHX}}(D,m) = \big(\check{\mathcal{B}}^{(1)}_{\mathrm{IHX}}(D,m)\big)_{G^\star_D}. \tag{3.13}$$

Then Eq. (3.10) follows from Eq. (3.11) in view of the relations (3.12) and (3.13). □

Let us introduce a notation $\mathcal{B}_m(D) = \widetilde{\mathcal{B}}_m(D)/\widetilde{\mathcal{B}}^{(1)}_{\mathrm{IHX}}(D,m)$.

Corollary 3.9. *The map $\widehat{A}$ establishes the isomorphism between the spaces $\mathcal{B}_m(D)$ and $\mathcal{H}^\star_m(D)$ (see Eq.* (2.9) *for the definition of the latter space).*

Proof. This isomorphism follows from the combination of Lemmas 3.5 and 3.7. □

We leave it for the reader to check that the isomorphism $\widehat{A}$ intertwines the maps

$$\begin{aligned} &\mathcal{B}_{m_1}(D_1) \otimes \mathcal{B}_{m_2}(D_2) \to \mathcal{B}_{m_1+m_2}(D_1 \cup D_2),\\ &\mathcal{H}^\star_{m_1}(D_1) \otimes \mathcal{H}^\star_{m_2}(D_2) \to \mathcal{H}^\star_{m_1+m_2}(D_1 \cup D_2), \end{aligned} \tag{3.14}$$

which come from the multiplications in the algebras $\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{D}}$ as defined in Section 2.

Proof of Theorem 3.1. According the definition (3.3) of the space $\widetilde{\mathcal{B}}^{(\mathrm{r},0)}_{m,n}$,

$$\widetilde{\mathcal{B}}^{(\mathrm{r},0)}_{m,n+m} = \bigoplus_{D:\ \chi(D)=n} \mathcal{B}_m(D), \tag{3.15}$$

while by its definition

$$\widetilde{\mathcal{D}}^{\star}_{m,n} = \bigoplus_{D:\ \chi(D)=n} \mathcal{H}^{\star}_m(D). \tag{3.16}$$

It is easy to see that $\widehat{A}$ establishes the isomorphisms

$$\widehat{A}: \widetilde{\mathcal{B}}^{(0)}_{\mathrm{AS}} \cap \widetilde{\mathcal{B}}^{(\mathrm{r},0)}_{m,n+m} \to \widetilde{\mathcal{D}}^{\star}_{\mathrm{AS}} \cap \widetilde{\mathcal{D}}^{\star}_{m,n}, \qquad \widetilde{\mathcal{B}}^{(0)}_{\mathrm{IHX}} \cap \widetilde{\mathcal{B}}^{(\mathrm{r},0)}_{m,n+m} \to \widetilde{\mathcal{D}}^{\star}_{\mathrm{IHX}} \cap \widetilde{\mathcal{D}}^{\star}_{m,n}. \tag{3.17}$$

Then Eq. (3.2) follows from Eqs. (3.3) and (2.28) together with the isomorphism of Corollary 3.9. □

4. Rationality conjecture

Recall that Kontsevich integral of a knot $\mathcal{K} \in S^3$ is a sequence of vectors $I^{\mathcal{B}}_{m,n}(\mathcal{K}) \in \mathcal{B}_{m,n}$, $m \geqslant 0$, $n \geqslant m$, depending on the topological class of $\mathcal{K}$. The space $\mathcal{B}_{0,0}$ is 1-dimensional, its basis vector is the empty graph, so it can be naturally identified with $\mathbb{C}$. It is known that $I^{\mathcal{B}}_{0,0}(\mathcal{K}) = 1$.

We combine the vectors $I^{\mathcal{B}}_n(\mathcal{K})$ into a formal power series of a formal variable $\hbar$

$$I^{\mathcal{B}}(\mathcal{K};\hbar) = 1 + \sum_{\substack{m\geqslant 0,\ n\geqslant m\\ m+n\geqslant 1}} I^{\mathcal{B}}_{m,n}(\mathcal{K})\hbar^n \in \mathcal{B}. \tag{4.1}$$

Prior to formulating a conjecture about the structure of $I^{\mathcal{B}}(\mathcal{K};\hbar)$ we have to apply to it some transformations. First, we apply the wheeling map $\widehat{\Omega}: \mathcal{B} \to \mathcal{B}$, described in [3], in order to produce

$$\begin{aligned} I^{\Omega}(\mathcal{K};\hbar) &= \widehat{\Omega}\big(I^{\mathcal{B}}(\mathcal{K};\hbar)\big) \\ &= 1 + \sum_{\substack{m,n\geqslant 0\\ m+n\geqslant 1}} I^{\Omega}_{m,n}(\mathcal{K})\hbar^{m+n} \in \mathcal{D}, \quad I^{\Omega}_{m,n} \in \mathcal{B}_{m,n}. \end{aligned} \tag{4.2}$$

Then we apply the isomorphism $\widehat{A}$, which maps Kontsevich integral from $\mathcal{B}$ to $\mathcal{D}$. More precisely, we choose a set $\boldsymbol{D}$ of 3-valent graphs D such that each type of a graph (without distinguishing them by cyclic order at vertices) is represented there exactly once, and then we map $\mathcal{B}$ to $\mathcal{D}_{\boldsymbol{D}}$ as described at the end of Section 2. Thus we get

$$\begin{aligned} I^{\mathcal{D}}(\mathcal{K};\hbar) &= \widehat{A}\big(I^{\mathcal{B}}(\mathcal{K};\hbar)\big) \\ &= 1 + \sum_{\substack{m,n\geqslant 0\\ m+n\geqslant 1}} I^{\mathcal{D}}_{m,n}(\mathcal{K})\hbar^{m+n} \in \mathcal{D}, \quad I^{\mathcal{D}}_{m,n}(\mathcal{K}) \in \mathcal{D}_{m,n}. \end{aligned} \tag{4.3}$$

By using the algebra structure of $\mathcal{D}$ and manipulating the formal power series in $\hbar$ we can define the logarithm of Kontsevich integral

$$
\begin{aligned}
I^{(\log)}(\mathcal{K};\hbar) &= \log I^{\mathcal{D}}(\mathcal{K};\hbar) \\
&= \sum_{\substack{m,n\geqslant 0\\ m+n\geqslant 1}} I^{(\log)}_{m,n}(\mathcal{K})\hbar^{m+n} \in \mathcal{D}, \quad I^{(\log)}_{m,n}(\mathcal{K}) \in \mathcal{D}_{m,n},
\end{aligned} \tag{4.4}
$$

through the formula

$$
\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}. \tag{4.5}
$$

The advantage of the logarithm $I^{(\log)}(\mathcal{K};\hbar)$ is that it can be expressed exclusively in terms of *connected* 3-valent graphs.

Kontsevich integral $I^{(\log)}(\mathcal{K};\hbar)$ belongs to the quotient space (2.26). Let $\tilde{I}^{(\log)}(\mathcal{K};\hbar)$ be a representative of $I^{(\log)}(\mathcal{K};\hbar)$ in the space $\widetilde{\mathcal{D}}$ (Of course, it is defined only up to an element of $\widetilde{\mathcal{D}}_{\mathrm{IHX}}$). We present $\tilde{I}^{(\log)}(\mathcal{K};\hbar)$ as

$$
\tilde{I}^{(\log)}(\mathcal{K};\hbar) = \sum_{D\in \boldsymbol{D}_{\mathrm{c}}} \sum_{m=0}^{\infty} x_m(\mathcal{K}, D)\hbar^{\chi(D)+m}, \tag{4.6}
$$

where $\boldsymbol{D}_{\mathrm{c}} \subset \boldsymbol{D}$ is a subset of connected 3-valent graphs and $x_m(\mathcal{K}, D) \in \mathcal{H}_m(D)$.

Now we are almost ready to formulate our conjecture. Let V be a vector space. For $x \in V$ we define $e^x \in S^*V$ by the power series $e^x = \sum_{n=0}^{\infty} x^n/n!$. If Λ is a lattice in V, then we extend this exponential map to an injection of a group algebra $\mathrm{Exp}:\mathbb{Q}[\Lambda] \to S^*V$. For a graph D, $H^1(D,\mathbb{Z})$ forms a lattice in $H^1(D,\mathbb{Q})$. We denote

$$
H^{(\exp)}(D,\mathbb{Q}) = \mathrm{Exp}\big(\mathbb{Q}\big[H^1(D,\mathbb{Z})\big]\big)_{G_D} \subset \mathcal{H}(D). \tag{4.7}
$$

In other words, $H^{(\exp)}(D,\mathbb{Q})$ is G_D-invariant part of the rational span of the exponents of the elements of $H^1(D,\mathbb{Z})$ and Exp establishes its isomorphism with $(\mathbb{Q}[H^1(D,\mathbb{Z})])_{G_D}$.

Now recall that if D has N edges, then e_j $(1 \leqslant j \leqslant N)$ denote the oriented edges forming a basis in the space of 1-chains C_1, while f_j, $1 \leqslant j \leqslant N$, form the dual basis in the dual space C_1^*. In view of Eq. (3.8) we can think of f_j as elements of $H^1(D,\mathbb{Q})$.

Lemma 4.1. *The product of the Alexander–Conway polynomial of e^{f_j} is an element of the algebra $H^{(\exp)}(D,\mathbb{Q})$:*

$$
\prod_{j=1}^{N} \Delta_{\mathrm{A}}\big(\mathcal{K}; \exp(f_j)\big) \in H^{(\exp)}(D,\mathbb{Q}), \tag{4.8}
$$

and its inverse is a well-defined element of $\mathcal{H}(D)$.

Proof. To prove relation (4.8), we have to show that its l.h.s. is G_D-invariant. The elements of the group G_D not only permute f_j, $1 \leqslant j \leqslant N$, but they may also reverse the orientation of some edges of D and thus change the signs of corresponding f_j. However, the relation

$$
\Delta_{\mathrm{A}}(\mathcal{K}; 1/t) = \Delta_{\mathrm{A}}(\mathcal{K}; t), \tag{4.9}
$$

guarantees that this change of sign does not affect the expression (4.8) and hence it is G_D-invariant. At the same time, the Alexander–Conway polynomial satisfies the property $\Delta_{\mathrm{A}}(\mathcal{K};1)=1$ which guarantees that the inverse of (4.8) can be inverted within $\mathcal{H}(D)$. $\square$

Let us introduce a notation

$$I^{(\log)}(\mathcal{K},D)=\sum_{m=0}^{\infty}x_m(\mathcal{K},D)\in\big(S^*H^1(D,\mathbb{Q})\big)_{G_D}. \tag{4.10}$$

The only 3-valent graph D with $\chi(D)=0$ is a circle. The value of $I^{(\log)}(\mathcal{K},\text{circle})$ has been established by Bar-Natan and Garoufalidis in [2]

$$I^{(\log)}(\mathcal{K},\text{circle})=\frac{1}{2}\left[\log\left(\frac{\sinh(f/2)}{(f/2)}\right)-\log\Delta_{\mathrm{A}}\big(\mathcal{K};\exp(f)\big)\right], \tag{4.11}$$

where f represents the integral generator of $H^1(\text{circle},\mathbb{Q})$. Our conjecture deals with the value of $I^{(\log)}(\mathcal{K},D)$ for graphs with $\chi(D)\geqslant 1$. Recall that such graphs have exactly $N=3\chi(D)$ edges.

Conjecture 4.2. *The representative $\tilde{I}^{(\log)}(\mathcal{K};\hbar)\in\widetilde{\mathcal{D}}$ of Kontsevich integral $I^{(\log)}(\mathcal{K};\hbar)\in\mathcal{D}$ can be chosen in such a way that for any $D\in\boldsymbol{D}$, $\chi(D)\geqslant 1$, there exists an element $y(\mathcal{K},D)\in H^{(\exp)}(D,\mathbb{Q})$ such that*

$$I^{(\log)}(\mathcal{K},D)=\frac{y(\mathcal{K},D)}{\prod_{j=1}^{3\chi(D)}\Delta_{\mathrm{A}}(\mathcal{K};\exp(f_j))}. \tag{4.12}$$

Remark 4.3. Andrew Kricker has proved this conjecture in his paper [7].

Remark 4.4 D. Thurston presented arguments which show that if Conjecture 4.2 is true as it is formulated, then it should also be true if one defines $I^{\mathcal{D}}(\mathcal{K},\hbar)$ directly as an image of $I^{\mathcal{B}}(\mathcal{K},\hbar)$ under the isomorphism $\widehat{A}$ without applying the wheeling map $\widehat{\Omega}$ of Eq. (4.2).

Remark 4.5. It is convenient to introduce some other notations in relation to Eq. (4.12). Let $p(\mathcal{K},D)\in(\mathbb{Q}[H^1(D,\mathbb{Z})])_{G_D}$ be such that $\mathrm{Exp}(p(\mathcal{K},D))=y(\mathcal{K},D)$. Also, if we index the edges of D in such a way that $f_1,\dots,f_{\chi(D)+1}$ form a basis of $H^1(D,\mathbb{Z})$ and $H^1(D,\mathbb{Q})$, then we can write $I^{(\log)}(\mathcal{K},D)$ and $y(\mathcal{K},D)$ more explicitly as

$$I^{(\log)}(\mathcal{K},D)=I^{(\log)}(\mathcal{K},D;f_1,\dots,f_{\chi(D)+1}), \tag{4.13}$$

$$p(\mathcal{K},D)=p(\mathcal{K},D;f_1,\dots,f_{\chi(D)+1}), \tag{4.14}$$

$$y(\mathcal{K},D)=p(\mathcal{K},D;e^{f_1},\dots,e^{f_{\chi(D)+1}}), \tag{4.15}$$

where

$$I^{(\log)}(\mathcal{K},D;x_1,\dots,x_{\chi(D)+1})\in\mathbb{Q}\big[[x_1,\dots,x_{\chi(D)+1}]\big], \tag{4.16}$$

$$p(\mathcal{K},D;t_1,\dots,t_{\chi(D)+1})\in\mathbb{Q}\big[t_1^{\pm1},\dots,t_{\chi(D)+1}^{\pm1}\big]. \tag{4.17}$$

5. Rational structure of the Jones polynomial

There is a well-known relation between the Kontsevich integral and the colored Jones polynomial of a knot, so the rationality Conjecture 4.2 should manifests itself in the structure of the latter object. In fact, this manifestation observed in [9], served for us as evidence which led to the rationality conjecture. Another advantage in establishing a relation between Eq. (4.12) and the rational expansion of the Jones polynomial [9] is that at present it is much easier to calculate the colored Jones polynomial than Kontsevich integral. Therefore, working out the rational expansion of [9] is a practical way of finding the polynomials $y(\mathcal{K}, D)$ of Eq. (4.12).

Let us recall the exact relation between the Kontsevich integral and a colored Jones (or, more generally, HOMFLY) polynomial based on a simple Lie algebra $\mathfrak{g}$. We equip $\mathfrak{g}$ with the ad-invariant scalar product normalized in such a way that long roots have length $\sqrt{2}$ (this scalar product allows us to identify the dual space $\mathfrak{g}^*$ with $\mathfrak{g}$ itself). Let $\vec{\alpha} \in \mathfrak{h}$ be the hightest weight of a representation of $\mathfrak{g}$, shifted by $\vec{\rho}$ (which is half the sum of positive roots of $\mathfrak{g}$). Reshetikhin and Turaev associate to this data a polynomial $J_{\vec{\alpha}}(\mathcal{K}; q) \in \mathbb{Z}[q^{\pm 1/2}]$. If we substitute

$$q = e^{\hbar}, \tag{5.1}$$

then we can expand $J_{\vec{\alpha}}(\mathcal{K}; q)$ in power series of $\hbar$

$$J_{\vec{\alpha}}(\mathcal{K}; q) = \sum_{n=0}^{\infty} p_n(\mathcal{K}; \vec{\alpha}) \hbar^n, \tag{5.2}$$

whose coefficients $p_n(\mathcal{K}; \vec{\alpha})$ are polynomials of $\vec{\alpha}$. The same series (5.2) can be deduced from the value of Kontsevich integral.

The data $\mathfrak{g}, \vec{\alpha}$ defines an element in the dual space $\mathcal{B}^*$, which is called *the weight system*. We will define it in such a way that it will be suitable for application to $I^{\Omega}(\mathcal{K}; \hbar)$. The first steps in the definition of the weight systems are fairly standard. Let $\vec{x}_a$, $1 \leqslant a \leqslant \dim \mathfrak{g}$, be a basis of $\mathfrak{g}$. Define the structure constants f_{abc} by the relation

$$[\vec{x}_a, \vec{x}_b] = \sum_{c=1}^{\dim \mathfrak{g}} f_{ab}{}^{c} \vec{x}_c. \tag{5.3}$$

We can raise and lower the indices of $f_{ab}{}^{c}$ with the help of the metric tensor

$$h_{ab} = \vec{x}_a \cdot \vec{x}_b \tag{5.4}$$

and its inverse h^{ab}.

Let D be a $(1,3)$-valent graph, $\deg_1(D) = m$, $\deg_2(D) = n + m$. Suppose that if we strip off its legs, then we get a 3-valent graph D_0. Let us orient the edges of D_0 and assign orientation to the edges of D in such a way that it is compatible with the orientation of D_0 and legs are oriented in the direction from 1-valent vertex to 3-valent vertex. Next, we assign the tensors f to 3-valent vertices, assigning their indices to attached edges according to the cyclic ordering. We use the upper indices for the incoming edges and lower indices for the outgoing edges. Finally, we take the product of all tensors f assigned to 3-valent vertices, contract each pair of indices of f's along each internal edge, while contracting each index assigned to a leg with α_a ($\vec{\alpha} = \sum_{a=1}^{\dim \mathfrak{g}} \alpha_a \vec{x}_a$). Thus we get a Weyl group invariant homogeneous polynomial $w^{\Omega}(D, \vec{\alpha})$ of $\vec{\alpha}$ of degree m. It is easy to see that it does not depend on the choice of orientation of the edges of D_0. For a fixed weight $\vec{\alpha}$, w^{Ω} assignes a number to each $(1,3)$-valent graph, so $w^{\Omega} \in \widetilde{\mathcal{B}}^*$. In fact, due to the anti-symmetry of f and to the Jacobi identity, satisfied by the commutator (5.3), w^{Ω} annihilates the subspaces $\widetilde{\mathcal{B}}_{\mathrm{AS}}$ and $\widetilde{\mathcal{B}}_{\mathrm{IHX}}$ and therefore it can be projected to $\mathcal{B}^*$.

The usual way to proceed further is to convert $w^{\Omega}(D)$ as a Weyl group invariant polynomial on $\mathfrak{h}$ into an element of $(S^m \mathfrak{g})_{\mathfrak{g}}$, then use a PBW map to convert it into an element of $(U\mathfrak{g})_{\mathfrak{g}}$ and calculate the trace of that element in a $\mathfrak{g}$ module with the highest weight $\vec{\alpha} - \vec{\rho}$, thus obtaining another polynomial $w(D, \vec{\alpha})$ of $\vec{\alpha} \in \mathfrak{h}$ which is the standard weight of the graph D coming from $\mathfrak{h}$, or thinking of it as a function on all graphs D, w is a weight system on $\mathcal{B}$. Then the relation between the expansion (5.2) and Kontsevich integral is

$$J_{\vec{\alpha}}(\mathcal{K}; q) = d_{\vec{\alpha}} \left(1 + \sum_{\substack{m,n \geqslant 0 \\ m+n \geqslant 1}} w\bigl(I^{\mathcal{B}}_{m,n}(\mathcal{K}), \vec{\alpha}\bigr) \hbar^{m+n} \right), \tag{5.5}$$

where $d_{\vec{\alpha}}$ is the dimension of the representation of $\mathfrak{g}$ with the shifted highest weight $\vec{\alpha}$. However, as explained in [3], the wheeling map allows one to get the expansion (5.2) straight from the weight $w^{\Omega}(D, \vec{\alpha})$ without going through PBW map and calculating the trace:

$$J_{\vec{\alpha}}(\mathcal{K}; q) = d_{\vec{\alpha}} \left(1 + \sum_{\substack{m,n \geqslant 0 \\ m+n \geqslant 1}} w^{\Omega}\bigl(I^{\Omega}_{m,n}(\mathcal{K}), \vec{\alpha}\bigr) \hbar^{m+n} \right). \tag{5.6}$$

This is the formula that we will work with, because the weight function $w^{\Omega}(D, \vec{\alpha})$ is easy to transfer from $\mathcal{B}$ to $\mathcal{D}$. The inverse of the dual isomorphism map $\widehat{A}^*$ maps the weight system $w^{\Omega} \in \mathcal{B}^*$ into an element of $\mathcal{D}^*$, which we will call $w^{\mathcal{D}}$. In order to see how $w^{\mathcal{D}}$ acts on $\mathcal{D}$ we come back to the calculation of $w^{\Omega}(D, \vec{\alpha})$ and modify it.

Suppose that $\mathfrak{g}$ has $2k$ roots $\lambda_1, \ldots, \lambda_{2k}$. Let us index them in such a way that $\lambda_1, \ldots, \lambda_k$ are positive roots and $\lambda_1, \ldots, \lambda_r$ are simple roots, r being the rank of $\mathfrak{g}$.

For a root λ of $\mathfrak{g}$ let P_λ denote the operator projecting $\mathfrak{g}$ onto the root space $V_\lambda \subset \mathfrak{g}$. We also introduce an operator $P_{\mathfrak{h}}$, projecting $\mathfrak{g}$ onto $\mathfrak{h}$. Let us assign a root of $\mathfrak{g}$ or the Cartan subalgebra to each internal edge of D. Let $\widetilde{S}$ be a set of all such assignments. For an assignment $c \in \widetilde{S}$ we modify the contraction of indices of tensors f in the following way: if an internal edge carries an index a at the beginning and index b at the end, then instead of contracting them (that is, instead of setting $a = b$ and taking a sum over their values) we bring in an extra factor P^a_b, where P is the projector corresponding to the subspace assigned to that edge by c, and then contract the pairs of indices a and b independently. In other words, we project Lie algebras $\mathfrak{g}$ flowing along the internal edges of D onto root spaces and Cartan subalgebras. Let us denote the resulting number as $w^{\Omega}_c(D, \vec{\alpha})$. Since the sum of projectors $P_{\mathfrak{h}}$ and P_λ for all roots λ of $\mathfrak{g}$ is equal to the identity operator, then

$$w^{\Omega}(D, \vec{\alpha}) = \sum_{c \in \widetilde{S}} w^{\Omega}_c(D, \vec{\alpha}). \tag{5.7}$$

The sum in the r.h.s. of this equation can be simplified. Since $\vec{\alpha} \in \mathfrak{h}$, then

$$[\vec{\alpha}, \vec{y}] = (\vec{\alpha} \cdot \lambda)\vec{y} \quad \text{if } \vec{y} \in V_\lambda, \qquad [\vec{\alpha}, \vec{y}] = 0 \quad \text{if } \vec{y} \in \mathfrak{h}. \tag{5.8}$$

Therefore, $w_c^{\Omega}(D, \vec{\alpha}) = 0$ unless the following two conditions are met. First, c must assign the same projector to internal edges of D which correspond to the same edge of D_0. Second, there is a *compatibility requirement* at every 3-valent vertex: Cartan subalgebra can be assigned to at most one of its edges and the sum of the roots on incoming edges is equal to the sum of the roots on outgoing edges. Thus we can replace the set $\widetilde{\boldsymbol{S}}$ in Eq. (5.7) with the set $\boldsymbol{S}$ of 'compatible' assignments whose elements assign subspaces to the edges of D_0 in such a way that the compatibility condition is satisfied at all of its vertices.

Eq. (5.8) also indicate that the effect of leg contractions is easy to take into account in the calculation of $w_c^{\Omega}(D, \vec{\alpha})$. If a leg is attached to at least one edge, to which a Cartan subalgebra is assigned, then $w_c^{\Omega}(D, \vec{\alpha}) = 0$. Otherwise, if m_j legs are attached on the left side of an oriented edge e_j of D_0 to which a root λ is assigned, then they contribute a factor of $(\vec{\alpha} \cdot \lambda)^{m_j}$. Let $\lambda_{c(j)}$ denote the root of $\mathfrak{g}$ assigned by $c \in \boldsymbol{S}$ to the edge e_j of D_0. If c assigns $\mathfrak{h}$ to e_j, then we set $\lambda_{c(j)} = 0$. With these notations we see that

$$w_c^{\Omega}(D, \vec{\alpha}) = w_c(D_0) \prod_{j=1}^{N} (\vec{\alpha} \cdot \lambda_{c(j)})^{m_j}, \tag{5.9}$$

where $w_c(D_0) = w_c^{\Omega}(D_0, \vec{\alpha})$ (we had to introduce this new notation because the graph D_0 has no legs and as a result $w_c^{\Omega}(D_0, \vec{\alpha})$ does not depend on $\vec{\alpha}$). Note that in Eq. (5.9) we adopted a convention that $0^0 = 1$.

The isomorphism (3.7) completes the translation of $w_c^{\Omega}(D, \vec{\alpha})$ into the language of 3-valent graphs. For an assignment $c \in \boldsymbol{S}$ consider a linear combination of edges

$$e_{c,\vec{\alpha}} = \sum_{j=1}^{N} (\vec{\alpha} \cdot \lambda_{c(j)}) e_j \in C_1. \tag{5.10}$$

According to the compatibility condition satisfied by c, $e_{c,\vec{\alpha}} \in \ker \partial = H_1(D, \mathbb{Q})$. Therefore, we can evaluate an element $x \in S^* H^1(D, \mathbb{Q})$ on $e_{c,\vec{\alpha}}$ and get a number (or a formal series) $x(e_{c,\vec{\alpha}})$. Eqs. (3.7), (5.9) and (5.10) indicate that for an element $x \in \check{\mathcal{B}}_m(D_0)/\widetilde{\mathcal{B}}^{(1)}_{\mathrm{IHX}}(D_0, m)$,

$$w_c^{\Omega}(x, \vec{\alpha}) = w_c(D_0)\big(\widehat{A}x\big)(e_{c,\vec{\alpha}}). \tag{5.11}$$

Then, according to Eq. (5.7), after taking a sum over the assignments of $\boldsymbol{S}$, we come to the following relation: for any $x \in \widetilde{\mathcal{B}}_m(D_0)$,

$$w^{\Omega}(x, \vec{\alpha}) = w^{\mathcal{D}}\big(\widehat{A}x, \vec{\alpha}\big), \tag{5.12}$$

where

$$w^{\mathcal{D}}(y, \vec{\alpha}) = \sum_{c \in \boldsymbol{S}} w_c(D_0) y(e_{c,\vec{\alpha}}), \qquad y \in \mathcal{H}_m^{\star}(D_0). \tag{5.13}$$

Thus Eq. (5.13) defines the element $w^{\mathcal{D}} \in \mathcal{D}^*$ corresponding to $w^{\Omega} \in \mathcal{B}^*$.

Applying Eq. (5.12) to Eq. (5.6), we find that

$$J_{\vec{\alpha}}(\mathcal{K};q)=d_{\vec{\alpha}}\left(1+\sum_{\substack{m,n\geqslant 0\\ m+n\geqslant 1}} w^{\mathcal{D}}\big(I^{\mathcal{D}}_{m,n}(\mathcal{K}),\vec{\alpha}\big)\hbar^{m+n}\right). \tag{5.14}$$

It is easy to see that the weight system $w^{\mathcal{D}}$ behaves nicely under the multiplication of elements of $\mathcal{D}$: $w^{\mathcal{D}}(xy,\vec{\alpha})=w^{\mathcal{D}}(x,\vec{\alpha})w^{\mathcal{D}}(y,\vec{\alpha})$ for any $x,y\in\mathcal{D}$. Therefore the analog of Eq. (5.14) holds for the modified integral (4.4)

$$\log\big(J_{\vec{\alpha}}(\mathcal{K};q)/d_{\vec{\alpha}}\big)=\sum_{\substack{m,n\geqslant 0\\ m+n\geqslant 1}} w^{\mathcal{D}}\big(I^{(\log)}_{m,n}(\mathcal{K}),\vec{\alpha}\big)\hbar^{m+n}, \tag{5.15}$$

and for its representative (4.6) in the space $\widetilde{\mathcal{D}}$

$$\log\big(J_{\vec{\alpha}}(\mathcal{K};q)/d_{\vec{\alpha}}\big)=\sum_{D\in\boldsymbol{D}}\sum_{m=0}^{\infty} w^{\mathcal{D}}\big(x_m(\mathcal{K},D),\vec{\alpha}\big)\hbar^{\chi(D)+m}. \tag{5.16}$$

By using the formula (5.13) for the weight system, we can rewrite Eq. (5.16) as

$$\log\big(J_{\vec{\alpha}}(\mathcal{K};q)/d_{\vec{\alpha}}\big)=\sum_{D\in\boldsymbol{D}}\sum_{c\in\boldsymbol{S}}\sum_{m=0}^{\infty} w_c(D)x_m(\mathcal{K},D)(e_{c,\vec{\alpha}})\hbar^{\chi(D)+m}, \tag{5.17}$$

where $x_m(\mathcal{K},D)(e_{c,\vec{\alpha}})$ denotes the evaluation of the element $x_m(\mathcal{K},D)\in\mathcal{H}_m(D)$ on $e_{c,\vec{\alpha}}\in H_1(D,\mathbb{Q})$. According to Eq. (5.10), $e_{c,\vec{\alpha}}$ is a linear function of $\vec{\alpha}$, while $x_m(\mathcal{K},D)(e_{c,\vec{\alpha}})$ is the homogeneous polynomial of $e_{c,\vec{\alpha}}$ of degree m. Therefore, Eq. (5.17) can be further modified as

$$\begin{aligned}\log\big(J_{\vec{\alpha}}(\mathcal{K};q)/d_{\vec{\alpha}}\big) &= \sum_{D\in\boldsymbol{D}}\hbar^{\chi(D)}\sum_{c\in\boldsymbol{S}} w_c(D)\sum_{m=0}^{\infty}x_m(\mathcal{K},D)(e_{c,\hbar\vec{\alpha}})\\ &= \sum_{D\in\boldsymbol{D}}\hbar^{\chi(D)}\sum_{c\in\boldsymbol{S}} w_c(D)I^{(\log)}(\mathcal{K},D)(e_{c,\hbar\vec{\alpha}})\end{aligned} \tag{5.18}$$

the last line coming from Eq. (4.10). Since by the definition of the dual basis $f_j(e_i)=\delta_{ij}$, then according to Eq. (5.10), $f_j(e_{c,\hbar\vec{\alpha}})=\hbar(\vec{\alpha}\cdot\lambda_{c(j)})$ and as a result, in view of (5.1),

$$I^{(\log)}(\mathcal{K},D)(e_{c,\hbar\vec{\alpha}})=I^{(\log)}\big(\mathcal{K},D;\hbar(\vec{\alpha}\cdot\lambda_{c(1)}),\ldots,\hbar(\vec{\alpha}\cdot\lambda_{c(r)})\big) \tag{5.19}$$

(see Eq. (4.13)). Eq. (4.11) allows us to write the contribution of the '1-loop' graph ($\chi(D)=0$) explicitly. Assignments c simply put different roots on the circle, $w_c(\text{circle})=1$ and

$$\begin{aligned}&\sum_{c\in\boldsymbol{S}} w_c(\text{circle})I^{(\log)}\big(\mathcal{K},\text{circle};\hbar(\vec{\alpha}\cdot\lambda_{c(1)})\big)\\ &\quad=\sum_{j=1}^{k}\log\left(\frac{q^{(\vec{\alpha}\cdot\lambda_j)/2}-q^{-(\vec{\alpha}\cdot\lambda_j)/2}}{\hbar(\vec{\alpha}\cdot\lambda_j)}\right)-\sum_{j=1}^{k}\log\Delta_{\mathrm{A}}\big(\mathcal{K};q^{\vec{\alpha}\cdot\lambda_j}\big).\end{aligned} \tag{5.20}$$

Thus if we exponentiate both sides of Eq. (5.18) and use the formulas (5.19), (5.20) and the dimension formula

$$d_{\vec{\alpha}} = \prod_{j=1}^{k} \frac{\vec{\alpha}\cdot\lambda_j}{\vec{\rho}\cdot\lambda_j}, \tag{5.21}$$

then we find that

$$J_{\vec{\alpha}}(\mathcal{K};q) = \frac{d_{q,\vec{\alpha}}}{\Delta_{\mathfrak{g}}(\mathcal{K};q^{\vec{\alpha}\cdot\lambda_1},\dots,q^{\vec{\alpha}\cdot\lambda_r})}\; C_{q,\mathfrak{g}}\;\exp\left(\sum_{n=1}^{\infty} J_n^{(\log)}(\mathcal{K};\vec{\alpha})\hbar^n\right) \tag{5.22}$$

where

$$J_n^{(\log)}(\mathcal{K};\vec{\alpha}) = \sum_{D\in\boldsymbol{D},\,\chi(D)=n}\;\sum_{c\in\boldsymbol{S}} w_c(D)\, I^{(\log)}\big(\mathcal{K},D;\hbar(\vec{\alpha}\cdot\lambda_{c(1)}),\dots,\hbar(\vec{\alpha}\cdot\lambda_{c(r)})\big), \tag{5.23}$$

while

$$d_{q,\vec{\alpha}} = \prod_{j=1}^{k} \frac{q^{(\vec{\alpha}\cdot\lambda_j/2)} - q^{-(\vec{\alpha}\cdot\lambda_j)/2}}{q^{(\vec{\rho}\cdot\lambda_j/2)} - q^{-(\vec{\rho}\cdot\lambda_j)/2}} \tag{5.24}$$

is called the quantum dimension of the $\mathfrak{g}$-module with highest weight $\vec{\alpha}-\vec{\rho}$ and

$$C_{q,\mathfrak{g}} = \sum_{j=1}^{k} \log\left(\frac{q^{(\vec{\rho}\cdot\lambda_j)/2} - q^{-(\vec{\rho}\cdot\lambda_j)/2}}{\hbar(\vec{\alpha}\cdot\lambda_j)}\right) = 1 + \mathcal{O}\big(\hbar^2\big), \tag{5.25}$$

$$\Delta_{\mathfrak{g}}\big(\mathcal{K};q^{\vec{\alpha}\cdot\lambda_1},\dots,q^{\vec{\alpha}\cdot\lambda_r}\big) = \prod_{j=1}^{k} \Delta_{\mathrm{A}}\big(\mathcal{K};q^{\vec{\alpha}\cdot\lambda_k}\big). \tag{5.26}$$

Now let us apply Conjecture 4.2 to the r.h.s. of Eq. (5.23). According to Eqs. (4.12) and (4.13),

$$I^{(\log)}\big(\mathcal{K},D;\hbar(\vec{\alpha}\cdot\lambda_{c(1)}),\dots,\hbar(\vec{\alpha}\cdot\lambda_{c(r)})\big) = \frac{p(\mathcal{K},D;q^{\vec{\alpha}\cdot\lambda_{c(1)}},\dots,q^{\vec{\alpha}\cdot\lambda_{c(\chi(D)+1)}})}{\prod_{j=1}^{3\chi(D)} \Delta_{\mathrm{A}}(\mathcal{K};q^{\vec{\alpha}\cdot\lambda_{c(j)}})}. \tag{5.27}$$

Therefore if we bring all terms in the sums of Eq. (5.23) to the common denominator

$$\Delta_{\mathfrak{g}}^{3n}\big(\mathcal{K};q^{\vec{\alpha}\cdot\lambda_1},\dots,q^{\vec{\alpha}\cdot\lambda_r}\big),$$

then we find that $J_n^{(\log)}(\mathcal{K};\vec{\alpha})$ has a rational form

$$\begin{aligned} J_n^{(\log)}(\mathcal{K};\vec{\alpha}) &= \frac{p_n^{(\log)}(\mathcal{K};q^{\vec{\alpha}\cdot\lambda_1},\dots,q^{\vec{\alpha}\cdot\lambda_r})}{\Delta_{\mathfrak{g}}^{3n}(\mathcal{K};q^{\vec{\alpha}\cdot\lambda_1},\dots,q^{\vec{\alpha}\cdot\lambda_r})},\\ p_n^{(\log)}(\mathcal{K};t_1,\dots,t_r) &\in \mathbb{Q}\big[t_1^{\pm1},\dots,t_r^{\pm1}\big]. \end{aligned} \tag{5.28}$$

Then substituting this formula to Eq. (5.22), exponentiating the formal power series and expanding $C_{q,\mathfrak{g}}$ in powers of $\hbar$ we come to the following

Corollary of Conjecture 4.1. *For a knot $\mathcal{K}$ and a simple algebra $\mathfrak{g}$ there exist the polynomials*

$$p_n(\mathcal{K}; t_1, \ldots, t_r) \in \mathbb{Q}\big[t_1^{\pm 1}, \ldots, t_r^{\pm 1}\big], \quad n \geqslant 0, \tag{5.29}$$

such that

$$J_{\vec{\alpha}}(\mathcal{K}; q) = \frac{d_{q,\vec{\alpha}}}{\Delta_{\mathfrak{g}}(\mathcal{K}; q^{\vec{\alpha}\cdot\lambda_1}, \ldots, q^{\vec{\alpha}\cdot\lambda_r})} \times \left(1 + \sum_{n=1}^{\infty} \frac{p_n(\mathcal{K}; q^{\vec{\alpha}\cdot\lambda_1}, \ldots, q^{\vec{\alpha}\cdot\lambda_r})}{\Delta_{\mathfrak{g}}^{3n}(\mathcal{K}; q^{\vec{\alpha}\cdot\lambda_1}, \ldots, q^{\vec{\alpha}\cdot\lambda_r})} \hbar^n\right). \tag{5.30}$$

We can check this prediction for the case of $\mathfrak{g} = su(2)$. In fact, in this case the power of $\Delta_{\mathfrak{g}}$ in denominators (5.30) can be reduced. Indeed, the algebra $su(2)$ has only one positive root. As a result, the elements of $\boldsymbol{S}$ assign the subspaces of $su(2)$ to the edges of a graph D in such a way that for any three edges attached to the same vertex, two are assigned a root space and the third is assigned the Cartan subalgebra. Therefore, of $3\chi(D)$ edges that a graph D has, $\chi(D)$ edges always carry a Cartan subalgebra and only $2\chi(D)$ edges carry the root spaces. Therefore, in case of $su(2)$ Eq. (5.30) is reduced to

$$J_\alpha(\mathcal{K}; q) = \frac{[\alpha]}{\Delta_{\mathrm{A}}(\mathcal{K}; q^\alpha)} \left(1 + \sum_{n=1}^{\infty} \frac{p_n(\mathcal{K}; q^\alpha)}{\Delta_{\mathrm{A}}^{2n}(\mathcal{K}; q^\alpha)} \hbar^n\right), \tag{5.31}$$

where α is the dimension of the $su(2)$ module attached to the knot $\mathcal{K}$ and

$$[\alpha] = \frac{q^{\alpha/2} - q^{-\alpha/2}}{q^{1/2} - q^{-1/2}} \tag{5.32}$$

is its quantum dimension.

Eq. (5.31) can be verified directly. We proved in [9] that for a knot $\mathcal{K}$ in S^3 there exist the polynomials

$$P_n(\mathcal{K}; t) \in \mathbb{Z}\big[t^{\pm 1}\big], \quad n \geqslant 1, \tag{5.33}$$

such that the expansion (5.2) can be rewritten as

$$J_\alpha(\mathcal{K}; q) = \frac{[\alpha]}{\Delta_{\mathrm{A}}(\mathcal{K}; q^\alpha)} \left(1 + \sum_{n=1}^{\infty} \frac{P_n(\mathcal{K}; q^\alpha)}{\Delta_{\mathrm{A}}^{2n}(\mathcal{K}; q^\alpha)} h^n\right), \tag{5.34}$$

where

$$h = q - 1 = e^\hbar - 1. \tag{5.35}$$

It is easy to see that Eq. (5.31) follows easily from Eq. (5.34).

6. 2-loop invariant and the $SU(3)$ colored Jones polynomial

Let us describe more precisely the implications of Conjecture 4.2 for the value of Kontsevich integral at the level of '2-loop' graphs, i.e., the graphs with $\chi(D) = 1$. There are only 2 such connected graphs in $\boldsymbol{D}_{\mathrm{c}}$: the theta-graph D_1 and the dumbbell D_2 of Fig. 3. Therefore, we can present the 2-loop part of the Kontsevich integral (4.4) as

$$\sum_{m=0}^{\infty} I_{m,1}^{(\log)}(\mathcal{K}) = I^{(\log)}(\mathcal{K}, D_1;\, f_{1,D_1}, f_{2,D_1}) + I^{(\log)}(\mathcal{K}, D_2;\, f_{1,D_2}, f_{2,D_2}) \tag{6.1}$$

(cf. Eqs. (4.6), (4.10) and (4.13)), where we used a notation f_{i,D_j} instead of simply f_i in order to distinguish the dual edges coming from different graphs. The formal power series in the r.h.s. of Eq. (6.1) are not themselves the invariants of $\mathcal{K}$. They become the invariants only after the factorization over the subspace $\widetilde{\mathcal{D}}_{\mathrm{IHX}}$ (see Theorem 2.2 and preceding discussion). Let us describe the IHX indeterminacy in these power series more precisely. The graph D_3 of Fig. 3 is the only connected 2-loop graph with a 4-valent vertex. Applying the operator $\hat{\partial}_{\mathrm{IHX}}$ of (2.25) to an element $z(f_{1,D_3}, f_{2,D_3}) \in S^*H^1(D_3, \mathbb{Q})$ we get

$$\begin{aligned}
&\frac{2}{3}\big[z(f_{1,D_1}, f_{2,D_1}) + z(f_{2,D_1}, -f_{1,D_1} - f_{2,D_1}) + z(-f_{1,D_1} - f_{2,D_1}, f_{1,D_1})\big] \\
&\quad - z(f_{1,D_2}, f_{2,D_2}) \in \bigoplus_{i=1}^{2} \big(S^*H^1(D_i, \mathbb{Q})\big)_{G_i}.
\end{aligned} \tag{6.2}$$

In this formula we assumed for simplicity of notation that $z(x_1, x_2) \in \mathbb{Q}[[x_1, x_2]]$ already has the symmetries

$$z(x_1, x_2) = z(x_2, x_1) = -z(-x_1, x_2), \tag{6.3}$$

which makes the additional symmetrization of the expression (6.2) unnecessary. Expression (6.2) indicates that by using the IHX freedom we can bring the expression (6.1) to the form

$$\sum_{m=0}^{\infty} I_{m,1}^{(\log)}(\mathcal{K}) = I_\theta(\mathcal{K};\, f_{1,D_1}, f_{2,D_1}) \in \big(S^*H^1(D_1, \mathbb{Q})\big)_{G_{D_1}}, \tag{6.4}$$

where

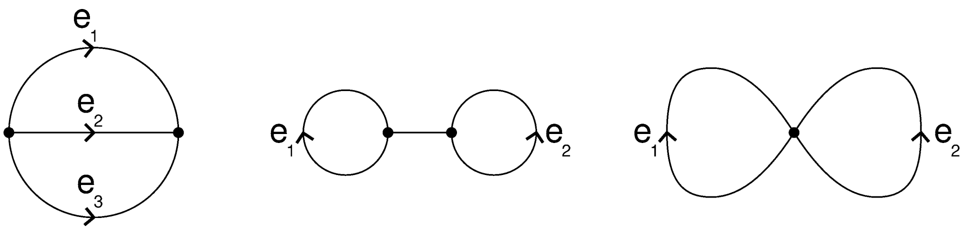

Fig. 3. The 2-loop graphs D_1, D_2 and D_3.

$$\begin{aligned} I_\theta(\mathcal{K};x_1,x_2) \;=\;& I^{(\log)}(\mathcal{K},D_1;x_1,x_2) \\ &+\frac{2}{3}\big[I^{(\log)}(\mathcal{K},D_2;x_1,x_2)+I^{(\log)}(\mathcal{K},D_2;x_2,-x_1-x_2) \\ &\quad +I^{(\log)}(\mathcal{K},D_2;-x_1-x_2,x_1)\big], \end{aligned} \tag{6.5}$$

thus eliminating the graph D_2 from Kontsevich integral. At the same time, expression (6.2) shows that $I_\theta(\mathcal{K};x_1,x_2)$ of Eq. (6.4) is the IHX-invariant combination and therefore it is the only 2-loop invariant of $\mathcal{K}$.

The rationality conjecture implies that $I_\theta(\mathcal{K};x_1,x_2)$ also has a rational structure. Indeed, according to the conjecture, one can use the IHX freedom in order to bring the terms in the r.h.s. of Eq. (6.1) to the following form:

$$\begin{aligned} I^{(\log)}(\mathcal{K},D_1;x_1,x_2) \;&=\; \frac{p(\mathcal{K},D_1;e^{x_1},e^{x_2})}{\Delta_{\mathrm{A}}(\mathcal{K};e^{x_1})\Delta_{\mathrm{A}}(\mathcal{K};e^{x_2})\Delta_{\mathrm{A}}(\mathcal{K};e^{x_1+x_2})} \\ I^{(\log)}(\mathcal{K},D_2;x_1,x_2) \;&=\; \frac{p(\mathcal{K},D_2;e^{x_1},e^{x_2})}{\Delta_{\mathrm{A}}(\mathcal{K};e^{x_1})\Delta_{\mathrm{A}}(\mathcal{K};e^{x_2})}. \end{aligned} \tag{6.6}$$

Then according to Eq. (6.5), $I_\theta(\mathcal{K};x_1,x_2)$ has a form

$$I_\theta(\mathcal{K};x_1,x_2)=\frac{p_\theta(\mathcal{K};e^{x_1},e^{x_2})}{\Delta_{\mathrm{A}}(\mathcal{K};e^{x_1})\Delta_{\mathrm{A}}(\mathcal{K};e^{x_2})\Delta_{\mathrm{A}}(\mathcal{K};e^{x_1+x_2})}, \tag{6.7}$$

where the polynomial $p_\theta(\mathcal{K};t_1,t_2)\in\mathbb{Q}[t_1^{\pm1},t_2^{\pm1}]$ is an invariant of $\mathcal{K}$. Both this polynomial and a rational function

$$I^*_\theta(\mathcal{K};t_1,t_2)=\frac{p_\theta(\mathcal{K};t_1,t_2)}{\Delta_{\mathrm{A}}(\mathcal{K};t_1)\Delta_{\mathrm{A}}(\mathcal{K};t_2)\Delta_{\mathrm{A}}(\mathcal{K};t_1t_2)} \tag{6.8}$$

have the symmetries

$$f(t_1,t_2)=f(t_2,t_1)=f\big((t_1t_2)^{-1},t_2\big)=f\big(t_1^{-1},t_2^{-1}\big) \tag{6.9}$$

implied by the symmetry group G_{D_1}. Finally, we rewrite Eq. (6.4) with the help of Eq. (6.7)

$$\begin{aligned} \sum_{m=0}^{\infty} I^{(\log)}_{m,1}(\mathcal{K}) \;&=\; I_\theta(\mathcal{K};f_{1,D_1},f_{2,D_1}) \\ &=\; \frac{p_\theta(\mathcal{K};e^{f_{1,D_1}},e^{f_{2,D_1}})}{\Delta_{\mathrm{A}}(\mathcal{K};e^{f_{1,D_1}})\Delta_{\mathrm{A}}(\mathcal{K};e^{f_{2,D_1}})\Delta_{\mathrm{A}}(\mathcal{K};e^{-f_{1,D_1}-f_{2,D_1}})}. \end{aligned} \tag{6.10}$$

It is easy to see from its definition that Kontsevich integral (4.1) does not contain $(1,3)$-valent graphs without legs. The wheeling map $\widehat{\Omega}$ produces such graphs, however their Euler characteristic is at least 2. Therefore, $I^{(\log)}_{0,1}(\mathcal{K})=0$ in (4.4) and this means that

$$I_\theta(\mathcal{K};0,0)=I^*_\theta(\mathcal{K};1,1)=0. \tag{6.11}$$

The polynomial $p_\theta(\mathcal{K};t_1,t_2)$ can be extracted from the colored $SU(3)$ Jones polynomial as described in Section 5. In [11] we will prove a slightly strengthened version of Eq. (5.30) for the groups $SU(n)$:

$$J_{\vec{\alpha}}(\mathcal{K};q) = \frac{d_{q,\vec{\alpha}}}{\Delta_{\mathfrak{g}}(\mathcal{K};q^{\vec{\alpha}\cdot\lambda_1},\dots,q^{\vec{\alpha}\cdot\lambda_r})} \times \left(1+\sum_{n=1}^{\infty}\frac{P_n(\mathcal{K};q^{\vec{\alpha}\cdot\lambda_1},\dots,q^{\vec{\alpha}\cdot\lambda_r})}{\Delta_{\mathfrak{g}}^{3n}(\mathcal{K};q^{\vec{\alpha}\cdot\lambda_1},\dots,q^{\vec{\alpha}\cdot\lambda_r})}h^n\right), \tag{6.12}$$

$$P_n(\mathcal{K};t_1,\dots,t_r) \in \mathbb{Z}\big[t_1^{\pm 1},\dots,t_r^{\pm 1}\big]$$

(note that here we used an expansion parameter $h = e^{\hbar} - 1$ instead of $\hbar$ and as a result obtained the polynomials with *integer* coefficients). For the case of $SU(3)$ this formula implies that

$$J_{\vec{\alpha}}(\mathcal{K};q) = \frac{d_{q,\vec{\alpha}}}{\Delta_{\mathfrak{g}}(\mathcal{K};q^{\vec{\alpha}\cdot\lambda_1},q^{\vec{\alpha}\cdot\lambda_2})}\big(1+\hbar F_1\big(\mathcal{K};q^{\vec{\alpha}\cdot\lambda_1},q^{\vec{\alpha}\cdot\lambda_2}\big)+\mathcal{O}\big(\hbar^2\big)\big), \tag{6.13}$$

where

$$F_1(\mathcal{K};t_1,t_2) = \frac{P_1(\mathcal{K};t_1,t_2)}{[\Delta_{\mathrm{A}}(\mathcal{K};t_1)\Delta_{\mathrm{A}}(\mathcal{K};t_2)\Delta_{\mathrm{A}}(\mathcal{K};t_1t_2)]^3}. \tag{6.14}$$

As we explained in Section 5, a similar formula (5.22) can be obtained by applying the $su(3)$ weight system to the logarithm of the Kontsevich integral of $\mathcal{K}$. Comparing Eqs. (6.13) and (5.22) and taking into account that $C_{q,\mathfrak{g}} = 1+\mathcal{O}(\hbar^2)$, we see that

$$F_1\big(\mathcal{K};q^{\vec{\alpha}\cdot\lambda_1},q^{\vec{\alpha}\cdot\lambda_2}\big) = J_1^{(\log)}(\mathcal{K};\vec{\alpha}). \tag{6.15}$$

Eqs. (5.23), (6.8) and (6.10) show that

$$J_1^{(\log)}(\mathcal{K};\vec{\alpha}) = \sum_{c\in \boldsymbol{S}} w_c(D_1) I_\theta^*\big(\mathcal{K};q^{\vec{\alpha}\cdot\lambda_{c(1)}},q^{\vec{\alpha}\cdot\lambda_{c(2)}}\big) \tag{6.16}$$

and the sum in this formula goes over the compatible assignments of root spaces and Cartan subalgebra to the egdes of the θ-shaped graph D_1. There are two types of such assignments. The first one assigns two opposite roots to two edges and Cartan subalgebra to the third edge, so $w_c(D_1) = 2$. There are 3 choices of pairs of roots, and within each choice there are 6 distinct assignments which give the same contributions due to the symmetries (6.9). Therefore, the total contribution of the first assignment to the r.h.s. of Eq. (6.16) is

$$12\big(I_\theta^*\big(\mathcal{K};q^{\vec{\alpha}\cdot\lambda_{c(1)}},1\big)+I_\theta^*\big(\mathcal{K};q^{\vec{\alpha}\cdot\lambda_{c(2)}},1\big)+I_\theta\big(\mathcal{K};q^{\vec{\alpha}\cdot(\lambda_1+\lambda_2)},1\big)\big). \tag{6.17}$$

Assignments of the second type put 3 different roots on the edges of D_1, so $w_c(D_1) = 1$ There are 2 choices of compatible triplets of roots, and there are 6 ways to assign each triplet to the egdes of D_1. Thus we have 12 assignments of the second type, and each gives the same contribution due to the symmetries (6.9). Therefore, the total contribution of the second assignment to the r.h.s. of Eq. (6.16) is

$$12 I_\theta^*\big(\mathcal{K};q^{\vec{\alpha}\cdot\lambda_1},q^{\vec{\alpha}\cdot\lambda_2}\big). \tag{6.18}$$

Thus putting the sum of (6.17) and (6.18) in the r.h.s. of Eq. (6.16) we find from Eq. (6.15) that

$$F_1(\mathcal{K};t_1,t_2) = 12\big(I^*_\theta(\mathcal{K};t_1,1)+I^*_\theta(\mathcal{K};t_2,1) + I^*_\theta\big(\mathcal{K};(t_1t_2)^{-1},1\big)+I^*_\theta(\mathcal{K};t_1,t_2)\big). \tag{6.19}$$

It is easy to solve this equation for $I^*_\theta(\mathcal{K};t_1,t_2)$. By setting $t_2=1$ and using Eq. (6.11) and the symmetries (6.9) we get

$$F_1(\mathcal{K};t_1,1) = 36 I^*_\theta(\mathcal{K};t_1,1), \tag{6.20}$$

hence

$$I^*_\theta(\mathcal{K};t_1,t_2) = \frac{1}{36}\big(3F_1(\mathcal{K};t_1,t_2)-F_1(\mathcal{K};t_1,1) - F_1(\mathcal{K};t_2,1)-F_1\big(\mathcal{K};(t_1t_2)^{-1},1\big)\big). \tag{6.21}$$

In [11] we will present a relatively efficient way of calculating $F_1(\mathcal{K};t_1,t_2)$. We have already written a Maple V program [12] which implements this algorithm. For a knot presented as a cyclic closure of a braid, this program calculates $\Delta_{\mathrm{A}}(\mathcal{K};t)$, $P_1(\mathcal{K};t_1,t_2)$ of Eq. (6.14) and then it finds $p_\theta(\mathcal{K};t_1,t_2)$ through Eq. (6.21).

7. Discussion

Since the first version of this paper was written, Kricker [7] has proved Conjecture 4.2. In fact, he proved it for a more general case of knots in integer homology spheres, where an analog of Kontsevich integral for knots is defined with the help of the LMO invariant [8] or its Århus version [4]. This knot invariant lies in the same space $\mathcal{B}$, so the previous discussion equally applies in that case. The analog of the colored Jones polynomial is the so-called *trivial connection contribution to the colored Jones polynomial* defined for $SU(2)$ in [10] for knots in rational homology spheres. It also has a rational structure (5.34).

Naturally, one wants to extend the rationality conjecture to the most general case of links in rational homology spheres. Unfortunately, Kricker's proof works only for integer homology spheres, so it cannot be generalized easily to homologically non-trivial knots in rational homology spheres, for which the analog of the rationality conjecture can be easily formulated in accordance with the $SU(2)$ results of [10]—one just has to use fractional exponents $e^{f_i/h(\mathcal{K})}$ in Eq. (4.15), where $h(\mathcal{K})$ is the order of the homology element represented by the knot $\mathcal{K}$.

Generalizing the rationality conjecture to links is not a straightforward exercise, because the arguments of Section 3 hinge upon Lemma 3.3. For this lemma to work, the legs of a $(1,3)$-valent graph have to be interchangeable (or, in other words, 'commutative'). In case of links however, legs are attached to different components, and as a result, one may have a non-zero graph in $\mathcal{B}$ which has two legs attached to the same 3-valent vertex, if they come from different link components.

Garoufalidis and Kricker [5] have circumvented this difficulty in the case of boundary links and proved an analog of the rationality property. However, the rationality property of the $SU(2)$ colored Jones polynomial of links described in [10] suggests a different approach. Namely, there is a graph space map which sends Kontsevich integral of a link into a close relative of the space $\mathcal{D}$. This map is similar to the Århus map [4]. It implements diagrammatically the stationary phase integration performed in [10] and ultimately it 'makes' all legs commutative. Similarly to the Århus map, one would have to prove that the image of the map is a link invariant, and this is work in progress.

Despite the fact that polynomials (4.17) share the variables $t_1, \ldots, t_{\chi(D)+1}$ with the Alexander polynomial, their topological interpretation remains unclear. First of all, because of the IHX indeterminacy, the rational expressions (4.12) are not knot invariants. Only their linear combinations which are insensitive to the IHX transformations are true invariants of knots. We explained this point in details in Section 6 for 2-loop graphs. In that case we presented an explicit linear combination (6.5) which is invariant and which yields a 2-loop invariant polynomial $p_\theta(\mathcal{K}; t_1, t_2)$. So just as a beginning, it would be interesting to establish its topological interpretation.

In the framework of the quantum Chern–Simons field theory and in the framework of the theory of finite type (Vassiliev) invariants, $I_\theta(\mathcal{K}; x_1, x_2)$ and the polynomial $p_\theta(\mathcal{K}; t_1, t_2)$ are analogs of the Casson–Walker invariant of rational homology spheres, so one might try to related $p_\theta(\mathcal{K}; t_1, t_2)$ to the moduli space of flat connections in the knot complement for an appropriate Lie group. At a simpler ('1-loop') level the Alexander polynomial $\Delta_{\mathrm{A}}(\mathcal{K}; t)$ is the analog of the order of integer homology $H_1(M; \mathbb{Z})$ of a rational homology sphere. The order of $H_1(M; \mathbb{Z})$ is equal to the number of flat $U(1)$ connections on M. At the same time, at least for fibered knots, $\Delta_{\mathrm{A}}(\mathcal{K}; t)$ is related to the monodromy map acting on the moduli space $\mathcal{M}_{U(1)}(\Sigma)$ of flat $U(1)$ connections on the Seifert surface Σ of $\mathcal{K}$. Namely, the monodromy map $f : \Sigma \to \Sigma$ which defines the structure of a fiber bundle for the knot complement $S^3 \setminus \mathcal{K}$, acts also on $\mathcal{M}_{U(1)}(\Sigma)$ and

$$\Delta_{\mathrm{A}}(\mathcal{K}; t) = \sum_{n=0}^{2g(\Sigma)} (-1)^n t^{n-g(\Sigma)} \operatorname{Tr}_{H^n(\mathcal{M}_{U(1)}(\Sigma))} f^*, \tag{7.1}$$

where f^* denotes the action of f on $H^n(\mathcal{M}_{U(1)}(\Sigma))$. Since the Casson–Walker invariant 'counts' the number of flat $SU(2)$ connections on a rational homology sphere, then one might expect that $p_\theta(\mathcal{K}; t_1, t_2)$ can be expressed somehow similarly to Eq. (7.1) through the action of the monodromy f on moduli spaces of flat connections of other Lie groups. Unfortunately, no such interpretation exists at present.

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Appendix A. The 2-loop polynomial $p_\theta(\mathcal{K}; t_1, t_2)$ for knots with up to 8 crossing

Here are the results of calculating the polynomials $p_\theta(\mathcal{K}; t_1, t_2)$ for the first few knots (with up to 8 crossings). We present these results in two different ways. First, as we

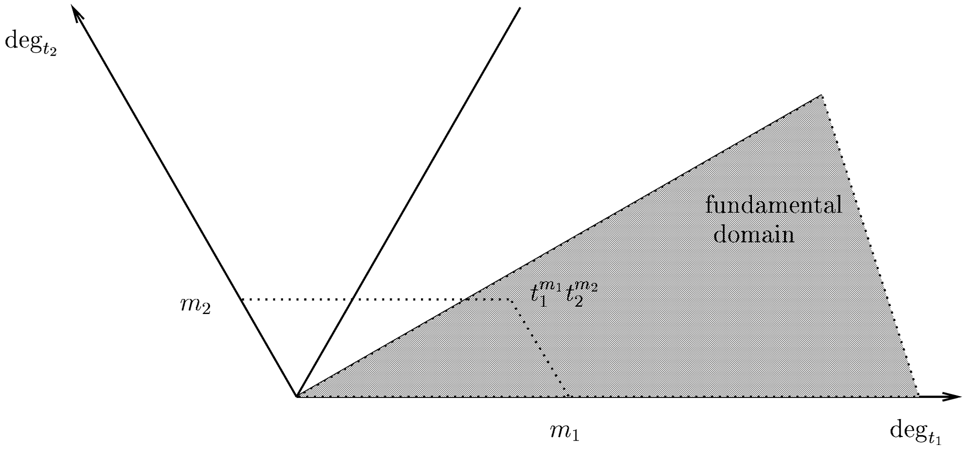

Fig. 4. Fundamental domain of the symmetry (6.9).

know, $p_\theta(\mathcal{K}) \in (\mathbb{Q}[H^1(D_1, \mathbb{Z})])_{G_{D_1}}$ and relations (6.9) come from the symmetry G_{D_1}. More explicitly, $H^1(D_1, \mathbb{Z})$ looks like $su(3)$ root lattice with elements f_1 and f_2 (and variables t_1, t_2) corresponding to the simple roots (see Fig. 4). The symmetry group G_{D_1} is the symmetry of this lattice (which preserves the origin). So instead of writing the whole polynomial $p_\theta(\mathcal{K}; t_1, t_2)$ we may list just the monomials belonging to a fundamental domain of G_{D_1}. From our $su(3)$ lattice description it is easy to see that we may choose a fundamental domain to include the monomials

$$t_1^{m_1} t_2^{m_2}, \quad m_1, m_2 \geqslant 0,\ m_1 \geqslant 2m_2. \tag{A.1}$$

Then the other monomials will be determined by the symmetries (6.9). Similarly, in view of the symmetry (4.9) it is enough to list only the monomials of $\Delta_{\mathrm{A}}(\mathcal{K}; t)$ with non-negative powers of t. Thus in Table 1 we present the 'fundamental domain' parts of the Alexander polynomial $\Delta_{\mathrm{A}}(\mathcal{K}; t)$ and (scaled) 2-loop polynomial $12 p_\theta(\mathcal{K}; t_1, t_2)$.

An alternative way of describing $p_\theta(\mathcal{K}; t_1, t_2)$ comes from the observation that the ring of Laurent polynomials with the symmetries (6.9) can be written as $\mathbb{Q}[u_1, u_2]$, where

$$\begin{aligned} u_1(t_1, t_2) &= t_1 + t_1^{-1} + t_2 + t_2^{-1} + t_1 t_2 + t_1^{-1} t_2^{-1}, \\ u_2(t_1, t_2) &= t_1^2 t_2 + t_1^{-2} t_2^{-1} + t_1 t_2^2 + t_1^{-1} t_2^{-2} + t_1 t_2^{-1} + t_1^{-1} t_2. \end{aligned} \tag{A.2}$$

So in Table 2 we present the expressions for the Alexander polynomial $\Delta_{\mathrm{A}}(\mathcal{K}; t)$ in terms of $u = t + t^{-1}$ and for the (scaled) 2-loop polynomial $12 p_\theta(\mathcal{K}; t_1, t_2)$ in terms of u_1 and u_2.

Remark A.1. If $\mathcal{K}'$ is the mirror image of $\mathcal{K}$, then $p_\theta(\mathcal{K}'; t_1, t_2) = -p_\theta(\mathcal{K}; t_1, t_2)$, hence $p_\theta(\mathcal{K}; t_1, t_2) = 0$ for amphicheiral knots.

Remark A.2. As we see, experimental evidence suggests that

$$12 p_\theta(\mathcal{K}; t_1, t_2) \in \mathbb{Z}\big[t_1^{\pm 1}, t_2^{\pm 1}\big]. \tag{A.3}$$

Table 1
The Alexander polynomial $\Delta_{\mathrm{A}}(\mathcal{K};t)$ and the 2-loop polynomial $12p_\theta(\mathcal{K};t_1,t_2)$ presented by monomials in funamental domains

Knot	$\Delta_{\mathrm{A}}(\mathcal{K};t)$	$12p_\theta(\mathcal{K};t_1,t_2)$
3_1	$t-1$	$-t_1^2t_2+t_1^2$
4_1	t^2-3t+5	0
5_1	t^2-t+1	$2t_1^4t_2^2-2t_1^4t_2+2t_1^4-t_1^2t_2+t_1^2$
5_2	$2t-3$	$-13t_1^2t_2+9t_1^2+6t_1-12$
6_1	$-2t+5$	$3t_1^2t_2-t_1^2-6t_1+24$
6_2	$-t^2+3t-3$	$-3t_1^4t_2^2+3t_1^4t_2-t_1^4-6t_1^3-11t_1^2t_2+15t_1^2$
6_3	t^2-3t+5	0
7_1	t^3-t^2+t-1	$-3t_1^6t_2^3+3t_1^6t_2^2-3t_1^6t_2+2t_1^4t_2^2+3t_1^6-2t_1^4t_2+2t_1^4-t_1^2t_2+t_1^2$
7_2	$3t-5$	$-58t_1^2t_2+36t_1^2+36t_1-84$
7_3	$2t^2-3t+3$	$-25t_1^4t_2^2+25t_1^4t_2-17t_1^4+7t_1^2t_2-12t_1^3+t_1^2-6t_1+12$
7_4	$4t-7$	$136t_1^2t_2-80t_1^2-96t_1+240$
7_5	$2t^2-4t+5$	$41t_1^4t_2^2-33t_1^4t_2-16t_1^3t_2+17t_1^4+12t_1^2t_2+32t_1^3+4t_1^2-14t_1+36$
7_6	$-t^2+5t-7$	$-7t_1^4t_2^2+5t_1^4t_2+10t_1^3t_2-t_1^4-20t_1^3-98t_1^2t_2+64t_1^2+50t_1-108$
7_7	t^2-5t+9	$-5t_1^2t_2+t_1^2+12t_1-48$
8_1	$-3t+7$	$23t_1^2t_2-9t_1^2-36t_1+132$
8_2	$-t^3+3t^2-3t$	$6t_1^6t_2^3-6t_1^6t_2^2+6t_1^6t_2+20t_1^4t_2^2-2t_1^6-20t_1^4t_2-12t_1^5+30t_1^4$
	$+3$	$-11t_1^2t_2-6t_1^3+15t_1^2$
8_3	$-4t+9$	0
8_4	$-t^2+3t-3$	$-3t_1^4t_2^2+3t_1^4t_2-t_1^4-11t_1^2t_2-6t_1^3+15t_1^2$
8_5	$-t^3+3t^2-4t$	$-10t_1^6t_2^3+8t_1^6t_2^2+6t_1^5t_2^2-6t_1^6t_2-29t_1^4t_2^2-6t_1^5t_2+2t_1^6+12t_1^4t_2$
	$+5$	$+12t_1^5+13t_1^3t_2-15t_1^4+15t_1^2t_2+6t_1^3-43t_1^2+16t_1-48$
8_6	$-2t^2+6t-7$	$-31t_1^4t_2^2+27t_1^4t_2+12t_1^3t_2-9t_1^4-111t_1^2t_2-54t_1^3+111t_1^2+18t_1-12$
8_7	t^3-3t^2+5t-5	$5t_1^6t_2^3-5t_1^6t_2^2+3t_1^6t_2-t_1^4t_2^2+6t_1^5t_2-t_1^6-7t_1^4t_2-6t_1^5$
		$+4t_1^3t_2-3t_1^4+19t_1^2t_2+16t_1^3-31t_1^2$
8_8	$2t^2-6t+9$	$-5t_1^4t_2^2+3t_1^4t_2+6t_1^3t_2-t_1^4-5t_1^2t_2-6t_1^3-9t_1^2+18t_1-60$
8_9	$-t^3+3t^2-5t+7$	0
8_{10}	t^3-3t^2+6t-7	$7t_1^6t_2^3-6t_1^6t_2^2-3t_1^5t_2^2+3t_1^6t_2-2t_1^4t_2^2+9t_1^5t_2-t_1^6+2t_1^4t_2$
		$-6t_1^5-5t_1^3t_2-14t_1^4+48t_1^2t_2+20t_1^3-40t_1^2-18t_1+36$
8_{11}	$-2t^2+7t-9$	$-39t_1^4t_2^2+31t_1^4t_2+28t_1^3t_2-9t_1^4-206t_1^2t_2-76t_1^3+160t_1^2$
		$+74t_1-132$
8_{12}	$t^2-7t+13$	0
8_{13}	$2t^2-7t+11$	$-5t_1^4t_2^2+3t_1^4t_2+6t_1^3t_2-t_1^4-7t_1^2t_2-6t_1^3-9t_1^2+24t_1-84$
8_{14}	$-2t^2+8t-11$	$-47t_1^4t_2^2+35t_1^4t_2+48t_1^3t_2-9t_1^4-356t_1^2t_2-102t_1^3+236t_1^2$
		$+168t_1-336$

Table 2
The Alexander polynomial $\Delta_{\mathrm{A}}(\mathcal{K};t)$ and the 2-loop polynomial $12p_\theta(\mathcal{K};t_1,t_2)$ expressed in terms of symmetric polynomials u and u_1, u_2

Knot	$\Delta_{\mathrm{A}}(\mathcal{K};t)$	$12p_\theta(\mathcal{K};t_1,t_2)$
8_{15}	$3t^2-8t+11$	$203t_1^4t_2-148t_1^4t_2-145t_1^3t_2+57t_1^4+375t_1^2t_2+240t_1^3$ $-111t_1^2-304t_1+756$
8_{16}	t^3-4t^2+8t-9	$-9t_1^6t_2^3+8t_1^6t_2^2+4t_1^5t_2^2-4t_1^6t_2-26t_1^4t_2^2-16t_1^5t_2+t_1^6$ $+30t_1^4t_2+12t_1^5+7t_1^3t_2+6t_1^4-90t_1^2t_2-66t_1^3+106t_1^2+6t_1$ $+12$
8_{17}	$-t^3+4t^2-8t+11$	0
8_{18}	$-t^3+5t^2-10t+13$	0
3_1	$u-1$	$u_1^2-3u_2-2u_1-6$
4_1	u^2-3u+3	0
5_1	u^2-u-1	$2u_1^4-10u_2u_1^2-4u_1^3+10u_2^2+10u_2u_1-23u_1^2+53u_2+26u_1$ $+66$
5_2	$2u-3$	$9u_1^2-31u_2-12u_1-66$
6_1	$-2u+5$	$-u_1^2+5u_2-4u_1+30$
6_2	$-u^2+3u-1$	$-u_1^4+7u_2u_1^2-11u_2^2-5u_2u_1+31u_1^2-73u_2-34u_1-114$
6_3	u^2-3u+3	0
7_1	u^3-u^2-2u+1	$3u_1^6-21u_2u_1^4-6u_1^5+42u_2^2u_1^2+21u_1u_1^3-52u_1^4-21u_2^3$ $+215u_2u_1^2+62u_1^3-152u_2^2-16u_2u_1+268u_1^2-358u_2$ $-64u_1-276$
7_2	$3u-5$	$36u_1^2-130u_2-36u_1-300$
7_3	$2u^2-3u-1$	$-17u_1^4+93u_2u_1^2+38u_1^3-109u_2^2-121u_2u_1+221u_1^2-559u_2$ $-314u_1-702$
7_4	$4u-7$	$-80u_1^2+296u_2+64u_1+720$
7_5	$2u^2-4u+1$	$17u_1^4-101u_2u_1^2-50u_1^3+141u_2^2+165u_2u_1-200u_1^2+624u_2$ $+392u_1+672$
7_6	$-u^2+5u-5$	$-u_1^4+9u_2u_1^2-6u_1^3-19u_2^2+17u_2u_1+64u_1^2-194u_2-16u_1$ -324
7_7	u^2-5u+7	$u_1^2-7u_2+10u_1-54$
8_1	$-3u+7$	$-ou_1^2+41u_2-18u_1+186$
8_2	$-u^3+3u^2-3$	$-2u_1^6+18u_2u_1^4-48u_2^2u_1^2-6u_2u_1^3+66u_1^4+34u_2^3+12u_2^2u_1$ $-314u_2u_1^2-54u_1^3+324u_2^2+114u_2u_1-463u_1^2+977u_2$ $+248u_1+894$
8_3	$-4u+9$	0
8_4	$-u^2+3u-1$	$-u_1^4+7u_2u_1^2-11u_2^2-5u_2u_1+31u_1^2-73u_2-34u_1-114$
8_5	$-u^3+3u^2-u-1$	$2u_1^6-18u_2u_1^4-4u_1^5+50u_2^2u_1^2+32u_2u_1^3-59u_1^4-42u_2^3$ $-52u_2^2u_1+288u_2u_1^2+132u_1^3-359u_2^2-341u_2u_1+351u_1^2$ $-983u_2-528u_1-834$
8_6	$-2u^2+6u-3$	$-9u_1^4+63u_2u_1^2+8u_1^3-103u_2^2-57u_2u_1+231u_1^2-631u_2$ $-288u_1-822$

Table 2 (*continued*)

Knot	$\Delta_{\mathrm{A}}(\mathcal{K};t)$	$12p_\theta(\mathcal{K};t_1,t_2)$
8_7	u^3-3u^2+2u+1	$-u_1^6+u_2u_1^4+4u_1^5-26u_2^2u_1^2-23u_2u_1^3+11u_1^4+23u_2^3$ $+34u_2^2u_1-94u_2u_1^2-34u_1^3+157u_2^2+133u_2u_1-92u_1^2$ $+346u_2+112u_1+276$
8_8	$2u^2-6u+5$	$-u_1^4+7u_2u_1^2+4u_1^3-13u_2^2-11u_2u_1-5u_1^2-15u_2+8u_1$ $+6$
8_9	$-u^3+3u^2-2u+1$	0
8_{10}	u^3-3u^2+3u-1	$-u_1^6+9u_2u_1^4+6u_1^5-27u_2^2u_1^2-36u_2u_1^3+4u_1^4+27u_2^3$ $+54u_2^2u_1-62u_2u_1^2-58u_1^3+152u_2^2+198u_2u_1-9u_1^2$ $+259u_2+156u_1+138$
8_{11}	$-2u^2+7u-5$	$-9u_1^4+67u_2u_1^2+2u_1^3-119u_2^2-35u_2u_1+256u_1^2-730u_2$ $-260u_1-972$
8_{12}	$u^2-7u+11$	0
8_{13}	$2u^2-7u+7$	$-u_1^4+7u_2u_1^2+4u_1^3-13u_2^2-11u_2u_1-5u_1^2-17u_2+14u_1$ -18
8_{14}	$-2u^2+8u-7$	$-9u_1^4+71u_2u_1^2-8u_1^3-135u_2^2+3u_2u_1+300u_1^2-916u_2$ $-204u_1-1272$
8_{15}	$3u^2-8u+5$	$57u_1^4-376u_2u_1^2-166u_1^3+613u_2^2+569u_2u_1-687u_1^2$ $+2543u_2+1258u_1+2574$
8_{16}	u^3-4u^2+5u-1	$u_1^6-10u_2u_1^4-4u_1^5+33u_2^2u_1^2+25u_2u_1^3-12u_1^4-35u_2^3$ $-43u_2^2u_1+128u_2u_1^2+22u_1^3-244u_2^2-158u_2u_1+189u_1^2$ $-607u_2-182u_1-558$
8_{17}	$-u^3+4u^2-5u+3$	0
8_{18}	$-u^3+5u^2-7u+3$	0

Remark A.3. The degree of the Alexander polynomial is bounded by the genus of the knot $g(\mathcal{K})$

$$\deg \Delta_{\mathrm{A}}(\mathcal{K};t) \leqslant g(\mathcal{K}), \tag{A.4}$$

In view of the symmetries (6.9) (which come from G_{D_1}), the reasonable measure of the degree of $p_\theta(\mathcal{K};t_1,t_2)$ is the t_1 degree of its fundamental domain part. Let us denote it simply as $\deg p_\theta(\mathcal{K};t_1,t_2)$. Then Table 1 suggests a similar inequality

$$\deg p_\theta(\mathcal{K};t_1,t_2) \leqslant 2g(\mathcal{K}). \tag{A.5}$$

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